Non-classical continua, pseudobalance, and the law of action and reaction

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April 2, 2013

Dedicated to the memory of Luiz-Carlos Martins

Abstract The procedure followed in constructing models for generalized continua is revisited. It is shown that the microforce balance equations required for the description of generalized continua are not in fact expressions of the balance of physical quantities, but consequences of the regularity assumed for the systems of contact actions. In the proposed approach the law of action and reaction, which in classical continuum mechanics is a consequence of Euler's balance laws, recovers the status of a basic principle held in Newtonian mechanics. Some examples of generalized continua taken from the literature are discussed.

1 Introduction

In a recent revisitation[1] of the *method of virtual power*, one of the conclusions was that the two terms which form the equation of the virtual power, the external and the internal power, should not be the object of independent assumptions. Once an expression of the external power is assumed, the internal power can be *deduced*, using some regularity properties of the system of contact actions plus the indifference of the external power.

In the present paper, the roles of regularity and indifference have been separated. Body forces and surface tractions are supposed to be *measures*,

*My thanks to the anonymous reviewers, whose accurate and keen remarks contributed to the overall quality of the paper.

[1](Del Piero 2009).

that is, vector-valued set functions additive on disjoint subsets, which we call μ and Q, respectively. The surface tractions form a system of contact actions on the interior surfaces of the body, called a *Cauchy flux*.[2] The regularity assumptions are made on this system, while the indifference of power provides relations between Q and μ. This is the reason for separating the roles of regularity and indifference.

An appropriate set of regularity assumptions on Q, which defines what we call a *bounded Cauchy flux*, provides the general structure required for a self-consistent formulation of Continuum Mechanics. The first assumption is that the restriction of Q to the collection of the subsurfaces of the boundary of each fixed subbody be an absolutely continuous measure with respect to the area measure. The second is that the restriction of Q to the collection of the boundaries of all subbodies be a measure F absolutely continuous with respect to the volume measure. That is, Q has a surface density s, and F has a volume density f. Our third, and last, assumption is that Q is *skew-symmetric*. Thus, Q is defined on oriented surfaces S, and the sign of $Q(S)$ changes with the orientation of S. In mechanical terms, this corresponds to Newton's law of action and reaction.

The existence of a volume density f for F determines an integral relation, which we call a *pseudobalance equation*, between the densities f and s. This equation allows us to prove the dependence of s on the normal to the surface element and the linearity of this dependence. This can be done using two basic tools of continuum mechanics, Noll's theorem on the dependence of s on the normal and Cauchy's tetrahedron theorem. We emphasize that the existence of the Cauchy stress is not deduced, as usual, from the balance law of linear momentum, but from regularity assumptions on the system of contact actions.[3]

Only at this point, indifference enters the play. It is known that the balance laws of linear and angular momentum can be deduced from the indifference of the external power.[4] The first balance law has the same form as the pseudobalance equation, but is of a different nature: as a consequence of indifference, it is a relation between Q and the body force measure μ, and not between surface and volume densities of the same Cauchy flux Q. A comparison of the two equations leads to the identification of the volume density f of F with the volume density b of the body force. In this way the equation of virtual power is obtained, and from it a weak formulation of the equilibrium problem can be deduced.

[2] (Gurtin & Martins 1976).

[3] This is indeed the main idea in the paper (Gurtin & Martins 1976).

[4] (Noll 1963).

It may look awkward to invent a complicated procedure to end up with no new conclusions. In fact, the advantages of the new procedure become evident when it is extended to more general classes of continua. Essentially, there are two ways to generalize the classical definition of a continuum:

- *to relax the regularity assumed for Q and μ, by admitting the presence of singular measures,*
- *to consider additional external actions, represented by additional measures Q^α, μ^α.*

Here we consider generalizations of the second type, which are a standard way to define continua with microstructure. Our basic assumption is that not only Q, but also all Q^α are bounded Cauchy fluxes. As a consequence, with each Q^α is associated a pseudobalance equation relating the surface density σ^α to a volume density ϕ^α. But, while Q is related to the body force measure μ by the balance law of linear momentum, no such relation is assumed to hold between the measures Q^α and μ^α.

In the literature, the role of the pseudobalance equations is played by *microforce balance equations* which are either postulated, or deduced from an assumed expression of the internal power.[5] In both cases it is not clear whether or not these equations are considered as fundamental laws of mechanics, like the balance of linear momentum. If this were the case, a proliferation of postulates would take place, depending on the number and nature of the additional variables. Moreover, in most cases the new postulates would not be based on sound physical motivations.

This is the point where the pseudobalance equations play a basic role. They provide a proof of the existence of counterparts of the stress tensor for the Cauchy fluxes Q^α, without introducing extra balance laws. This makes possible the transformation of the external power into a volume integral, the *internal power*. The equality between external and internal power is the *equation of virtual power*. Note that this equation is not a relation between independent powers, as it is classically considered, but only an identity between equivalent representations of the same power. It turns out that this identity is all that is needed for the formulation of the equilibrium problem.

The paper can be divided into three parts. In the first part, Sections 2 to 4, some basic concepts of measure theory and geometric measure theory are

[5] For the first choice see the book (Capriz 1989), in which a law of *balance of micromomentum* is systematically used. On the second choice is based the method of virtual power developed in (Germain 1973b).

recalled. The reader not interested in technical details may skip this part.[6] Continuous bodies are identified with normalized sets of finite perimeter,[7] and the external actions are limited to the pair (Q, μ). A proof of Noll's theorem appropriate to this context is given in Appendix A. From it, the pseudobalance equation (4.17) is deduced.

In the second part, Sections 5 and 6, the traditional formulation of continuum mechanics and some alternative approaches are analyzed and compared with the approach based on bounded Cauchy fluxes. The final part, Sections 7 to 11, deals with continua with microstructure. For such continua, general forms of the pseudobalance equations and of the equation of virtual power are given in Section 7. The particular forms taken for specific continua depend on the *order parameters* which define the microstructure, and on the restrictions due to the indifference of power, which also vary according to the nature of the continuum. We say that the order parameters define the *structural properties* of a continuum, not to be confused with the *constitutive properties*, which characterize specific classes of materials, and which are not considered in this paper.

The structural properties and the indifference requirements determine a subdivision into classes of continua, some of which are discussed in the final Sections. Continua in which all measures Q^α and μ^α are indifferent to changes of observer are considered in Section 8, and *micropolar continua* and *Cosserat continua* are considered in Section 9. Section 10 deals with *second-gradient continua*. It includes some comments on the *edge forces* which show up when the contact forces are decomposed into a normal and a tangential part, as required for a correct formulation of the boundary conditions. Finally, Section 11 deals with *continua with latent microstructure*, characterized by the presence of internal constraints relating the order parameters to the macroscopic deformation.

2 Basic concepts and definitions

Let Ω be a bounded set in the N-dimensional Euclidean point space $\mathcal{E}^N$, and let $\wp(\Omega)$ be a collection of subsets of Ω, including Ω and the empty set $\emptyset$. We say that two elements Π_1, Π_2 of $\wp(\Omega)$ are *disjoint* if

$$\Pi \in \wp(\Omega)\,, \quad \Pi \subset \Pi_1\,, \quad \Pi \subset \Pi_2 \quad \Rightarrow \quad \Pi = \emptyset\,.$$

[6] For the reader who is interested in such details, a more general introduction can be found in (Vol'pert & Hudjaev 1985). Basic reference books are (Evans & Gariepy 1992) and (Ambrosio et al. 2000) for measure theory, and (Federer 1969) and (Ziemer 1989) for geometric measure theory.

[7] (Šilhavý 1991).

Assume that $\wp(\Omega)$ is equipped with a binary operation $\vee$, which with every pair (Π_1, Π_2) of elements of $\wp(\Omega)$ associates a set $(\Pi_1 \vee \Pi_2) \in \wp(\Omega)$, called the *join* of Π_1 and Π_2, such that

$$\begin{gathered} \Pi_1 \subset \Pi_1 \vee \Pi_2 \,, \qquad \Pi_2 \subset \Pi_1 \vee \Pi_2 \,, \\ \Pi_1 \subset \Pi \ \text{ and } \ \Pi_2 \subset \Pi \ \Rightarrow \ \Pi_1 \vee \Pi_2 \subset \Pi \,. \end{gathered} \tag{2.1}$$

Assume, further, that with each $\Pi \in \wp(\Omega)$ is associated a set $\Pi^c \in \wp(\Omega)$, called the *complement* of Π in Ω, such that

$$\Pi \text{ and } \Pi^c \text{ are disjoint}\,, \qquad \Pi \vee \Pi^c = \Omega \,. \tag{2.2}$$

The join and the complement are unique.[8] They define a second binary operation

$$\Pi_1 \wedge \Pi_2 \;=\; (\Pi_1^c \vee \Pi_2^c)^c, \tag{2.3}$$

called the *meet* of Π_1 and Π_2. Two sets in $\wp(\Omega)$ are disjoint if and only if their meet is the empty set.[9]

The existence of the join and of the complement provides $\wp(\Omega)$ with the structure of an *algebra of sets*. In particular, this structure includes the properties

$$\begin{array}{ll} \text{associative} & \Pi \vee (\Pi_1 \vee \Pi_2) \;=\; (\Pi \vee \Pi_1) \vee \Pi_2 \,, \\ \text{distributive} & \Pi \vee (\Pi_1 \wedge \Pi_2) \;=\; (\Pi \vee \Pi_1) \wedge (\Pi \vee \Pi_2) \,, \end{array}$$

plus the properties obtained by interchanging $\vee$ and $\wedge$ in the above relations. The set $\overline{\wp}(\Omega)$ of all countable joins and complements of elements of $\wp(\Omega)$ is the *σ-algebra* generated by $\wp(\Omega)$.

Let Y be a finite-dimensional inner-product space.[10] A *$Y-$ valued measure* on $\overline{\wp}(\Omega)$ is a map $\mu : \overline{\wp}(\Omega) \to Y$, additive on disjoint sets:

$$\Pi_1, \Pi_2 \in \overline{\wp}(\Omega)\,, \;\; \Pi_1 \wedge \Pi_2 = \emptyset \quad \Rightarrow \quad \mu(\Pi_1 \vee \Pi_2) = \mu(\Pi_1) + \mu(\Pi_2)\,.$$

Examples of real-valued measures are the N-dimensional Lebesgue measure $\mathcal{L}^N$ and the $(N-1)$-dimensional Hausdorff measure $\mathcal{H}^{N-1}$. For them, $\overline{\wp}(\Omega)$ is the set of all Lebesgue measurable subsets and of all subsets of $\mathcal{E}^N$, respectively. For both, Y is the real line, the join and the meet are the union

[8] A proof of this statement and of other statements made in this Section can be found in (Del Piero 2003), Sect. 5.

[9] The join and the meet are also called the *least envelope* and the *greatest common part* of Π_1 and Π_2, respectively ((Noll 1973), Sect. 8, (Truesdell 1991), Sect. 1.2).

[10] For example, the real line $\mathbb{R}$, or the set $\mathbb{R}^N$ of all vectors of dimension N, or the set $\mathbb{R}^{N\times N}$ of all linear transformations on $\mathbb{R}^N$ ($N\times N$ matrices, or second-order tensors).

and the intersection of sets, and the complement of Π is the complementary set $\Omega \setminus \Pi$. For $N = 3$, the two measures are called the *volume measure* and the *area measure*, respectively. For convenience we keep these names, and use the notations V, A in place of $\mathcal{L}^N, \mathcal{H}^{N-1}$, for every N.

A measure $\mu : \overline{\wp}(\Omega) \to Y$ is *absolutely continuous* with respect to the volume measure (to the area measure) if $\mu(\Pi) = 0$ for all $\Pi \in \overline{\wp}(\Omega)$ for which $V(\Pi) = 0$ (for which $A(\Pi) = 0$). A measure μ is said to be *singular* with respect to the same measures if there is a $\Pi_1 \in \overline{\wp}(\Omega)$ with $V(\Pi_1) = 0$ (with $A(\Pi_1) = 0$) such that $\mu(\Pi_2) = 0$ for all $\Pi_2 \in \overline{\wp}(\Omega)$ disjoint from Π_1. By the Lebesgue decomposition theorem, every measure μ admits a unique decomposition

$$\mu \,=\, \mu^a + \mu^s\,,$$

with μ^a absolutely continuous and μ^s singular with respect to the volume measure (to the area measure).

Let Π be a subset of $\mathcal{E}^N$, and let $B_r(x)$ be the N-dimensional ball with radius r centered at $x \in \mathcal{E}^N$. The limit

$$\rho(\Pi, x) \,=\, \lim_{r \to 0} \frac{V(B_r(x) \wedge \Pi)}{V(B_r(x))}\,, \tag{2.4}$$

if it exists, is a real number between 0 and 1. Then x is said to be a *point of density* for Π if $\rho(\Pi, x) = 1$, a *point of rarefaction* if $\rho(\Pi, x) = 0$, and a point of the *essential boundary* otherwise. The set of all density points, the set of all rarefaction points, and the essential boundary are also called the measure-theoretic interior, exterior, and boundary of Π. They will be denoted by Π^*, $\mathrm{ext}^*\Pi$, $\partial^*\Pi$, respectively. For the topological interior, exterior, and boundary of Π, the inclusions

$$\mathrm{int}\,\Pi \subset \Pi^*\,, \qquad \mathrm{ext}\,\Pi \subset \mathrm{ext}^*\,\Pi\,, \qquad \partial\Pi \supset \partial^*\Pi\,, \tag{2.5}$$

hold. It is also of interest that

$$\Pi^* = (\Pi^*)^*\,, \quad \mathrm{ext}^*\,\Pi = \mathrm{ext}^*\,(\Pi^*)\,, \quad \partial^*\Pi = \partial^*(\Pi^*)\,. \tag{2.6}$$

Let $H(x, n)$ be the half-space through x with exterior unit normal n. Set:

$$\rho(\Pi, x, n) \,=\, \lim_{r \to 0} \frac{V(B_r(x) \wedge \Pi \wedge H(x, n))}{V(B_r(x))}\,.$$

We say that a unit vector n is the measure-theoretic exterior normal, in short, the *exterior normal*, to Π at x if

$$\rho(\Pi, x, n) \,=\, \tfrac{1}{2}\,, \quad \rho(\Pi, x, -n) \,=\, 0\,.$$

The exterior normal, if it exists, is unique. At all points x at which the exterior normal exists, one has

$$\rho(\Pi, x) = \rho(\Pi, x, n) + \rho(\Pi, x, -n) \; = \; \tfrac{1}{2}\,.$$

Therefore, all such points belong to the essential boundary of Π.

The *perimeter* of a set is the area of the essential boundary. For all *sets of finite perimeter*, the following properties hold:

(i) *the finite unions and intersections and the complementary sets of sets of finite perimeter are sets of finite perimeter,*

(ii) *on the essential boundary of a set of finite perimeter the exterior normal exists A-almost everywhere,*[11]

(iii) *for a set Π of finite perimeter, Π and Π^* differ by a set of zero volume,*

(iv) *for a set Π of finite perimeter, the divergence theorem*

$$\int_{\partial^*\Pi} \varphi(x)\cdot\, n(x)\, dA \;=\; \int_{\Pi} \operatorname{div}\varphi(x)\, dV \tag{2.7}$$

holds for all Lipschitz continuous functions $\varphi : \mathbb{R}^N \to \mathbb{R}^N$.[12]

Note that, for

$$\varphi(x) = T^T(x)\, v(x)\,,$$

with T and v sufficiently regular tensor and vector fields, respectively, equation (2.7) provides the *Gauss-Green formula*

$$\int_{\partial^*\Pi} T(x)\, n(x)\cdot v(x)\, dA = \int_{\Pi} \big(T(x)\cdot\nabla v(x) + \operatorname{div}T(x)\cdot v(x)\big)\, dV\,. \tag{2.8}$$

A set Π is *normalized* if $\Pi = \Pi^*$. For normalized sets, it can be proved that the the rules (2.1), (2.2), and (2.3) are satisfied if and only if

$$\Pi_1 \vee \Pi_2 = (\Pi_1 \cup \Pi_2)^*\,, \quad \Pi^c = (\Omega \setminus \Pi)^*\,, \quad \Pi_1 \wedge \Pi_2 = \Pi_1 \cap \Pi_2\,. \tag{2.9}$$

That is, the join is the normalized union, the complement is the normalized complementary set, and the meet is the normalized intersection, which coincides with the intersection. Moreover, for normalized sets,

$$\Pi^c = \operatorname{ext}^*\Pi\,, \quad \operatorname{ext}^*(\Pi^c) = \Pi\,, \quad \partial^*(\Pi^c) = \partial^*\Pi\,. \tag{2.10}$$

[11] That is, except at most at a set of null area. See (Vol'pert & Hudjaev 1985), Section 4.5.3. In general, topological boundary and measure-theoretic boundary are not area-equivalent, see (Noll & Virga 1988) and ((Del Piero 2003), Section 3).

[12] (De Giorgi 1954; Federer 1958; Evans & Gariepy 1992). The regularity assumption on φ has been progressively relaxed, see footnote 26 below.

Let Ω be a normalized set of finite perimeter, and let $\wp(\Omega)$ be the set of all normalized subsets of Ω of finite perimeter. Because the join, the meet, and the complement of normalized sets are normalized sets, and the join, the meet, and the complement of sets of finite perimeter are sets of finite perimeter, $\wp(\Omega)$ is an algebra of sets. Its elements will be called the *subbodies* of Ω. Then, $\overline{\wp}(\Omega)$ is the σ-algebra generated by the subbodies of Ω.

3 Surfaces, oriented surfaces, surface measures

The *surface* of a subbody Π is the essential boundary $\partial^*\Pi$. The *subsurfaces* of $\partial^*\Pi$ are the intersections of a surface with other subbodies

$$S_1 = \partial^*\Pi \cap \Pi_1 \,, \qquad \Pi_1 \in \wp(\Omega) \,.$$

The relations

$$\partial^*\Pi_1 \cap \Pi_2 \;=\; \partial^*(\Pi_1 \cap \Pi_2) \cap \Pi_2 \;=\; \partial^*(\Pi_1 \vee \Pi_2^c) \cap \Pi_2 \tag{3.1}$$

hold for every pair of subbodies, and the relation

$$\partial^*\Pi_1 \cap (\Pi_1 \vee \Pi_2) \;=\; \partial^*\Pi_2 \cap (\Pi_1 \vee \Pi_2) \tag{3.2}$$

holds for every pair of disjoint subbodies. The proofs are left to the reader. The subsurface (3.2) is the *separating surface* of Π_1 and Π_2.

Let S be a subsurface of $\partial^*\Pi$. At a point x of S, consider the limit

$$\rho(S,x) = \lim_{r\to 0} \frac{A(S \cap B_r(x))}{A_r} \,,$$

where A_r is the area of the largest circle in $B_r(x)$. For $S = \partial^*\Pi$, this limit is equal to one for A-almost every $x \in \partial^*\Pi$.[13] Then for $S \subset \partial^*\Pi$ the same limit is a number between zero and one. We say that x is a *density point* for S if $\rho(S,x)=1$. The set of all density points of S will be denoted by $S^\star$.

A subsurface S is *normalized* if $S = S^\star$. The surfaces S and $S^\star$ differ at most by a set of area zero. If Π and Π_1 are normalized sets, the subsurface $S_1 = \partial^*\Pi \cap \Pi_1$ need not be normalized. An example is given in Fig. 1a, where the point H does not belong to Π_1 and, therefore, to S_1, but belongs to $S_1^\star$.

In a natural way, the subsurfaces of $\partial^*\Pi$ inherit the algebraic structure of $\wp(\Omega)$. For the normalized subsurfaces

$$S_1 = (\partial^*\Pi \cap \Pi_1)^\star \,, \quad S_2 = (\partial^*\Pi \cap \Pi_2)^\star \,, \tag{3.3}$$

[13] (Ziemer 1983), Sect. 2.

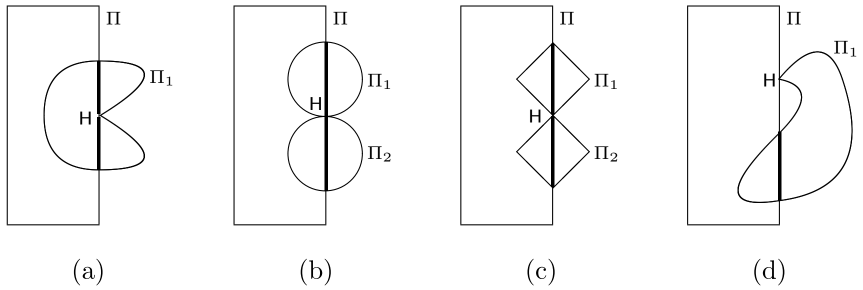

Figure 1. Two-dimensional examples of: a non-normalized subsurface $\partial^*\Pi \cap \Pi_1$ (a), two normalized subsurfaces $\partial^*\Pi \cap \Pi_1$, $\partial^*\Pi \cap \Pi_2$ for which $\partial^*\Pi \cap (\Pi_1 \vee \Pi_2)$ is a normalized subsurface, (b), and is a non-normalized subsurface, (c), and a normalized subsurface $\partial^*\Pi \cap \Pi_1$ for which $\partial^*\Pi \cap \Pi_1^c$ is a non-normalized surface (d)

the join, complement, and meet are defined by

$$\begin{aligned} S_1 \curlyvee S_2 &= (\partial^*\Pi \cap (\Pi_1 \vee \Pi_2))^\star\,, \qquad S_1^\Pi = (\partial^*\Pi \cap \Pi_1^c)^\star\,, \\ S_1 \curlywedge S_2 &= (\partial^*\Pi \cap \Pi_1 \cap \Pi_2)^\star\,, \end{aligned} \tag{3.4}$$

respectively. We say that S_1 and S_2 are disjoint if $S_1 \curlywedge S_2 = \emptyset$.

If Π, Π_1, Π_2 are normalized, the surfaces $(\partial^*\Pi \cap (\Pi_1 \vee \Pi_2))$, $(\partial^*\Pi \cap \Pi_1^c)$, $(\partial^*\Pi \cap \Pi_1 \cap \Pi_2)$ need not be normalized. For example, in both Fig. 1b and 1c the point H is a density point for $(\partial^*\Pi \cap (\Pi_1 \vee \Pi_2))$. But in the first case it belongs to $\Pi_1 \vee \Pi_2$ and in the second it does not. In Fig. 1d, H is a density point for $(\partial^*\Pi \cap \Pi_1^c)$ but does not belong to Π_1^c. Finally, for Π, Π_1 as in Fig. 1a and $\Pi_2 = \Omega$, $(\partial^*\Pi \cap \Pi_1 \cap \Pi_2)$ reduces to $\partial^*\Pi \cap \Pi_1$, which is not a normalized subsurface. This proves that the meet of subsurfaces does not coincide with the intersection.

Of fundamental importance are the following properties of the partition of a subbody Π into disjoint subbodies Π_1, Π_2, by means of a separating surface S.

Proposition 3.1. *Let Π, Π_1, Π_2 be subbodies such that*

$$\Pi = \Pi_1 \vee \Pi_2\,, \quad \Pi_1 \cap \Pi_2 = \emptyset\,, \tag{3.5}$$

let S be the normalized separating surface $S = (\partial^*\Pi_1 \cap \Pi)^\star$, and let

$$S_1 = (\partial^*\Pi_1 \cap \Pi_2^c)^\star\,, \quad S_2 = (\partial^*\Pi_2 \cap \Pi_1^c)^\star\,. \tag{3.6}$$

Then S, S_1, S_2 are pairwise disjoint, and

$$S_1 \curlyvee S_2 = \partial^*(\Pi_1 \vee \Pi_2)\,, \quad S \curlyvee S_1 = \partial^*\Pi_1\,, \quad S \curlyvee S_2 = \partial^*\Pi_2\,. \tag{3.7}$$

Proof. By (3.1),

$$S_1 = (\partial^*\Pi_1 \cap \Pi_2^c)^\star = (\partial^*\Pi \cap \Pi_2^c)^\star\,, \quad S_2 = (\partial^*\Pi_2 \cap \Pi_1^c)^\star = (\partial^*\Pi \cap \Pi_1^c)^\star\,.$$

Then both S_1 and S_2 are subsurfaces of $\partial^*\Pi$. By (3.4) and (2.3),

$$\begin{aligned} S_1 \curlywedge S_2 &= (\partial^*\Pi \cap \Pi_1^c \cap \Pi_2^c)^\star = (\partial^*\Pi \cap (\Pi_1 \vee \Pi_2)^c)^\star = (\partial^*\Pi \cap \Pi^c)^\star = \emptyset\,, \\ S_1 \curlyvee S_2 &= (\partial^*\Pi \cap (\Pi_1^c \vee \Pi_2^c))^\star = (\partial^*\Pi \cap (\Pi_1 \cap \Pi_2)^c)^\star = (\partial^*\Pi)^\star = \partial^*\Pi\,, \end{aligned}$$

with the second to last inequality due to the fact that $\Pi_1 \cap \Pi_2$ is the empty set, and therefore its complement is Ω. This proves $(3.7)_1$. Moreover, since both S and S_1 are subsurfaces of $\partial^*\Pi_1$, by (3.4) and by the distributive property,

$$\begin{aligned} S \curlywedge S_1 \;&=\; (\partial^*\Pi_1 \cap (\Pi_1 \vee \Pi_2) \cap \Pi_2^c)^\star \\ &=\; (\partial^*\Pi_1 \cap ((\Pi_1 \cap \Pi_2^c) \vee (\Pi_2 \cap \Pi_2^c)))^\star \;=\; (\partial^*\Pi_1 \cap \Pi_1 \cap \Pi_2^c)^\star \;=\; \emptyset\,, \end{aligned}$$

because both $\Pi_2 \cap \Pi_2^c$ and $\partial^*\Pi_1 \cap \Pi_1$ are empty. Moreover,

$$S \curlyvee S_1 \;=\; (\partial^*\Pi_1 \cap (\Pi_1 \vee \Pi_2 \vee \Pi_2^c))^\star \;=\; (\partial^*\Pi_1)^\star \;=\; \partial^*\Pi_1\,,$$

with the second equality due to $\Pi_2 \vee \Pi_2^c = \Omega$. To get $S \curlywedge S_2 = \emptyset$ and $(3.7)_3$ note that, by (3.2),

$$S = (\partial^*\Pi_1 \cap \Pi)^\star = (\partial^*\Pi_2 \cap \Pi)^\star\,,$$

that is, both S and S_2 are subsurfaces of $\partial^*\Pi_2$. Then it is sufficient to repeat the preceding proof, with the subscripts $_1$ and $_2$ interchanged. □

Another remarkable consequence of the relations (3.1) is the following decomposition of the essential boundary of an intersection.[14]

Proposition 3.2. *For every pair Π_1, Π_2 in $\wp(\Omega)$,*

$$\partial^*(\Pi_1 \cap \Pi_2) \;=\; (\partial^*\Pi_1 \cap \Pi_2)^\star \curlyvee (\partial^*\Pi_2 \cap \Pi_1)^\star\,. \tag{3.8}$$

Proof. By $(3.1)_1$,

$$\partial^*\Pi_1 \cap \Pi_2 = \partial^*(\Pi_1 \cap \Pi_2) \cap \Pi_2\,, \quad \partial^*\Pi_2 \cap \Pi_1 = \partial^*(\Pi_1 \cap \Pi_2) \cap \Pi_1\,.$$

[14] This Proposition generalizes a relation proved in (Ziemer 1983) for non-normalized surfaces, for which (3.8) holds only to within sets of null area.

Then, by (3.3) and $(3.4)_1$ with $\Pi = \Pi_1 \cap \Pi_2$,

$$(\partial^*\Pi_1 \cap \Pi_2)^\star \curlyvee (\partial^*\Pi_2 \cap \Pi_1)^\star = (\partial^*(\Pi_1 \cap \Pi_2) \,\cap\, (\Pi_2 \vee \Pi_1))^\star. \qquad (3.9)$$

On the other hand, by (3.3) and $(3.4)_2$ with $\Pi = \Pi_1 \cap \Pi_2$ and with $\Pi_1 \vee \Pi_2$ in place of Π_1,

$$((\partial^*(\Pi_1 \cap \Pi_2) \cap (\Pi_1 \vee \Pi_2))^\star)^\Pi = (\partial^*(\Pi_1 \cap \Pi_2) \,\cap\, (\Pi_1 \vee \Pi_2)^c)^\star.$$

The right-hand side is the empty set, because

$$\partial^*(\Pi_1 \cap \Pi_2) \,\subset\, \partial^*\Pi_1 \cup \partial^*\Pi_2\,, \qquad (\Pi_1 \vee \Pi_2)^c = \Pi_1^c \cap \Pi_2^c\,,$$

and both $\partial^*\Pi_1 \cap \Pi_1^c$ and $\partial^*\Pi_2 \cap \Pi_2^c$ are empty. If the complement of a subsurface of $\partial^*(\Pi_1 \cap \Pi_2)$ is empty, the subsurface coincides with the whole surface:

$$(\partial^*(\Pi_1 \cap \Pi_2) \cap (\Pi_1 \vee \Pi_2))^\star = \partial^*(\Pi_1 \cap \Pi_2)\,.$$

Then combining with (3.9) the desired relation (3.8) follows. □

The surface $\partial^*\Pi$ has a natural orientation, with the interior on the side of Π and the exterior on the side of Π^c. By $(2.10)_3$, $\partial^*\Pi$ can also be viewed as the surface of Π^c, and in this case it has the opposite orientation. The same holds for the subsurfaces $S_1 = (\partial^*\Pi \cap \Pi_1)^\star$. For a given orientation of $\partial^*\Pi$, with the symbol $\vec{S}$ we denote the subsurface S with the same orientation, and with $\overleftarrow{S}$ we denote the same subsurface with the opposite orientation.

By $\vec{\mathbb{S}}\,(\Pi)$ we denote the set of all countable joins and complements of normalized subsurfaces of $\partial^*\Pi$, oriented as $\partial^*\Pi$. This set is the σ-algebra generated by the subsurfaces of $\partial^*\Pi$. A Y-valued *surface measure* on $\partial^*\Pi$ is a function $Q : \vec{\mathbb{S}}\,(\Pi) \to Y$, additive on disjoint surfaces:

$$Q(S_1 \curlyvee S_2) = Q(S_1) + Q(S_2)\,, \qquad S_1, S_2 \in \vec{\mathbb{S}}\,(\Pi)\,, \;\; S_1 \curlywedge S_2 = \emptyset\,. \quad (3.10)$$

4 Cauchy fluxes and pseudobalance equations

Let $\overline{\mathbb{S}}\,(\Omega)$ be the set of all oriented surfaces in Ω:

$$\overline{\mathbb{S}}\,(\Omega) \;=\; \bigcup_{\Pi \in \overline{\wp}(\Omega)} \{\vec{S} \;:\; \vec{S} \in \vec{\mathbb{S}}\,(\Pi)\,\}\,.$$

A Y*-valued Cauchy flux* on Ω is a function $Q : \overline{\mathbb{S}}\,(\Omega) \to Y$, whose restriction to each $\vec{\mathbb{S}}\,(\Pi)$ is a surface measure on $\partial^*\Pi$. A Cauchy flux is *symmetric* if

$$Q(\vec{S}\,) \;=\; Q(\overleftarrow{S}\,) \qquad \forall \vec{S} \,\in\, \overline{\mathbb{S}}\,(\Omega)\,,$$

and is *skew-symmetric* if

$$Q(\vec{S}\,) \;=\; -Q(\overleftarrow{S}\,) \qquad \forall \vec{S} \in \overline{\mathbb{S}}(\Omega)\,.$$

An example of a real-valued symmetric Cauchy flux is the area measure. In the rest of the paper, we will be interested in skew-symmetric Cauchy fluxes. For them, the following additivity property for non-disjoint surfaces holds.[15]

Proposition 4.1. *For a Cauchy flux Q, the equation*

$$Q(\partial^*(\Pi_1 \vee \Pi_2)) \;=\; Q(\partial^*\Pi_1) + Q(\partial^*\Pi_2) \tag{4.1}$$

holds for all pairs Π_1, Π_2 of disjoint subbodies, if and only if Q is skew-symmetric.

Proof. Let $\vec{S}$ be the subsurface $(\partial^*\Pi_1 \cap (\Pi_1 \vee \Pi_2))^\star$ oriented as $\partial^*\Pi_1$, and let $\vec{S}_1, \vec{S}_2$ be the subsurfaces (3.6) oriented as $\partial^*\Pi_1$ and $\partial^*\Pi_2$, respectively. By Proposition 3.1 the three surfaces are pairwise disjoint, and equations (3.7), now rewritten as

$$\vec{S}_1 \curlyvee \vec{S}_2 \;=\; \partial^*(\Pi_1 \vee \Pi_2)\,, \quad \vec{S} \curlyvee \vec{S}_1 \;=\; \partial^*\Pi_1\,, \quad \overleftarrow{S} \curlyvee \vec{S}_2 \;=\; \partial^*\Pi_2\,,$$

hold. By the additivity property (3.10) on disjoint surfaces,

$$\begin{gathered} Q(\partial^*(\Pi_1 \vee \Pi_2)) \;=\; Q(\vec{S}_1) + Q(\vec{S}_2)\,, \\ Q(\partial^*\Pi_1) = Q(\vec{S}\,) + Q(\vec{S}_1)\,, \quad Q(\partial^*\Pi_2) = Q(\overleftarrow{S}\,) + Q(\vec{S}_2)\,. \end{gathered}$$

Then (4.1) holds if and only if $Q(\vec{S}\,) = -Q(\overleftarrow{S}\,)$. □

Let Q be a Cauchy flux, and let $F : \overline{\wp}(\Omega) \to Y$ be the function

$$F(\Pi) \;\doteq\; -Q(\partial^*\Pi)\,. \tag{4.2}$$

We are interested in whether or not F is a measure. The question is answered by the following

Proposition 4.2. *Let $Q : \overline{\mathbb{S}}(\Omega) \to Y$ be a Cauchy flux, and let F be as in* (4.2)*. Then F is a Y-valued measure on $\overline{\wp}(\Omega)$ if and only if Q is skew-symmetric.*

Proof. It is sufficient to prove that F is additive on disjoint subsets:

$$F(\Pi_1 \vee \Pi_2) = F(\Pi_1) + F(\Pi_2)\,, \qquad \Pi_1, \Pi_2 \in \overline{\wp}(\Omega)\,,\ \Pi_1 \cap \Pi_2 = \emptyset\,.$$

[15] (Noll 1973), Sect. 8.

By (4.2), this is the same as (4.1). By Proposition 4.1, equation (4.1) holds if and only if Q is skew-symmetric. $\square$

Thus, the restriction of a skew-symmetric Cauchy flux to the surfaces $\partial^*\Pi$ of all $\Pi \in \overline{\wp}(\Omega)$ can be identified with a measure F on $\overline{\wp}(\Omega)$. Consider the Lebesgue decomposition of F into the sum of an absolutely continuous and a singular part with respect to the volume measure

$$F = F^a + F^s .$$

By the Radon-Nikodým theorem,[16] F^a has a volume density $f \in L^1(\Omega, Y)$

$$F^a(\Pi) = \int_\Pi f(x)\, dV \qquad \forall\, \Pi \in \overline{\wp}(\Omega)\,, \tag{4.3}$$

while F^s is concentrated on a subset of Π with zero volume. Moreover, the Cauchy flux Q admits the decomposition $Q = Q^a + Q^s$, with

$$Q^a(\vec{S}\,) = \int_S s(x, \partial^*\Pi)\, dA \qquad \forall S \in \vec{\mathbb{S}}\,(\Pi)\,, \tag{4.4}$$

where $s(\,\cdot\,, \partial^*\Pi) \in L^1(\partial^*\Pi, Y)$ is the surface density associated with the restriction of Q^a to $\vec{\mathbb{S}}\,(\Pi)$, and Q^s is concentrated on a subset of $\partial^*\Pi$ with null area measure. Equation (4.2) then takes the form

$$\int_\Pi f(x)\, dV + F^s(\Pi) + \int_{\partial^*\Pi} s(x, \partial^*\Pi)\, dA + Q^s(\partial^*\Pi) \;=\; 0\,. \tag{4.5}$$

The measure Q is absolutely continuous with respect to the area measure if $Q^s = 0$, and F is absolutely continuous with respect to the volume measure if $F^s = 0$. A sufficient condition for the A-absolute continuity of Q is the property of *area boundedness*

$$|\, Q(\vec{S}\,)\,| \;\leq\; K\, A(\vec{S}\,)\,, \qquad \forall\, \vec{S} \;\in\; \overline{\vec{\mathbb{S}}}\,(\Omega)\,. \tag{4.6}$$

with K a positive constant and $|\cdot|$ the norm of Y. A sufficient condition for the V-absolute continuity of F is the *volume boundedness*

$$|\, F(\Pi)\,| \;\leq\; K\, V(\Pi) \qquad \forall\, \Pi \in \overline{\wp}(\Omega)\,. \tag{4.7}$$

The less restrictive condition that for every $\Pi \in \overline{\wp}(\Omega)$ there is a non-negative function $h^\Pi \in L^1(\partial^*\Pi, \mathbb{R})$ such that

$$|\, Q(\mathcal{S})\,| \;\leq\; \int_{\mathcal{S}} h^\Pi(x)\, dA \qquad \forall\, \mathcal{S} \in \vec{\mathbb{S}}\,(\Pi)\,, \tag{4.8}$$

[16] See e.g. (Ambrosio et al 2000).

is sufficient for the A-absolute continuity of Q, and the condition that there is a non-negative function $h \in L^1(\Omega, \mathbb{R})$ such that

$$| F(\Pi) | \leq \int_{\Pi} h(x) \, dV \qquad \forall \Pi \in \overline{\wp}(\Omega) \,, \tag{4.9}$$

is sufficient for the V-absolute continuity of F.[17] A flux with the properties (4.8) and (4.9) is called a *weakly balanced Cauchy flux*.[18]

Thus, F is a measure if and only if Q is skew-symmetric, and this measure is V-absolutely continuous if and only if Q is weakly balanced. A skew-symmetric weakly balanced Cauchy flux is called a *bounded Cauchy flux*.[19] For such fluxes, equation (4.2) takes the special form

$$\int_{\Pi} f(x) \, dV + \int_{\partial^* \Pi} s(x, \partial^* \Pi) \, dA \;=\; 0 \,. \tag{4.10}$$

Equation (4.5) and its special form (4.10) will be called *pseudobalance equations*, to distinguish them from the balance equations of continuum mechanics, which have the same form but a different physical meaning.

The pseudobalance equation (4.10) has two important consequences. For almost every $x \in \Omega$,

(i) *the surface density*

$$s(x, \partial^* \Pi) \;=\; \lim_{r \to 0} \frac{Q(\partial^* \Pi \cap B_r(x))}{A(\partial^* \Pi \cap B_r(x))} \tag{4.11}$$

only depends on the exterior normal n to $\partial^* \Pi$ at x,

$$s(x, \partial^* \Pi) = s(x, n) \qquad \text{a.e. } x \in \Omega \,, \tag{4.12}$$

(ii) *the dependence of s on n is linear.*

The first consequence was considered a postulate by Cauchy, and was proved later by Noll.[20] The second is the object of the celebrated *tetrahedron theorem* of Cauchy. Formally, they can be stated as follows.

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[17](Šilhavý 1985).
[18](Gurtin & Martins 1976), modified in (Šilhavý 1985).
[19](Šilhavý 2008).
[20](Noll 1959).

Theorem 4.3. *Let Q be a bounded Cauchy flux, and let Π be a subbody. Let x be a point of $\partial^*\Pi$ at which the measure-theoretic exterior normal n exists, and let $s(x, \partial^*\Pi)$ be the limit* (4.11). *Then for every other subbody Π' with $x \in \partial^*\Pi'$ and with the same normal n at x,*

$$s(x, \partial^*\Pi') \;=\; s(x, \partial^*\Pi)\,. \tag{4.13}$$

Theorem 4.4. *Let Q and s be as above, and let the function h^P in equation* (4.15) *below be integrable. Then there is a mapping $T \in L^1(\Omega, \mathbb{R}^{N\times N})$ such that*

$$s(x, n) = T(x)\, n \tag{4.14}$$

for all $n \in Y$ and for V-almost all x in Ω.

The proof of Theorem 4.3 given in Appendix A follows the lines of Noll's original proof,[21] with modifications dictated by the weaker regularity assumed for the surfaces $\partial^*\Pi$. For Theorem 4.4, the original proof based on the tetrahedron argument[22] is sufficient for the purpose of the present paper. It is known[23] that Cauchy's proof relies on the somehow artificial assumption of continuity of the vector field $s(\,\cdot\,, n)$. Later, the theorem was extended to integrable functions.[24]

A sufficient condition for integrability is the following. For every fixed direction n, let $\mathcal{P}^n_\xi$ be the plane with normal n and with signed distance ξ from a fixed point x_0. For an absolutely continuous flux Q, the function $s(\,\cdot\,, n)$ is integrable on each $\mathcal{P}^n_\xi$. Then, there is a non-negative number h^P, depending on ξ, such that

$$\int_{\mathcal{P}^n_\xi \cap \Omega} |s(x, n)|\, dA \;\le\; h^P(\xi)\, A(P^n_\xi \cap \Omega)\,. \tag{4.15}$$

Assume that the function $\xi \mapsto h^P(\xi)$ is integrable over the real line. If Ω is bounded, this guarantees the integrability of $s(\,\cdot\,, n)$ over Ω.[25]

For a bounded Cauchy flux, using (4.14) and the tensorial version of the divergence theorem (2.7)

$$\int_{\partial^*\Pi} T(x)\, n\, dA = \int_\Pi \mathrm{div} T(x)\, dV, \tag{4.16}$$

[21] (Noll 1959).
[22] (Cauchy 1823). For proofs not based on the tetrahedron argument see (Šilhavý 1985; 1990; 1991; 2008), (Fosdick & Virga 1989), (Marzocchi & Musesti 2003).
[23] (Noll 1973), Sect. 8.
[24] (Gurtin et al. 1968).
[25] (Kolmogorov & Fomin 1970), Sect. 35, Problem 6.

which follows from (2.8) written for a constant v, the pseudobalance equation (4.10) takes the form of a volume integral

$$\int_{\Pi} (f(x) + \mathrm{div} T(x))\, dV = 0\,. \tag{4.17}$$

By the arbitrariness of Π, this implies the local relation

$$f(x) + \mathrm{div} T(x) = 0 \qquad \text{a.e. } x \in \Omega\,, \tag{4.18}$$

between the divergence of T and the volume density of the measure F.[26]

5 The traditional approach to continuum mechanics

In this Section we recall the traditional formulation of continuum mechanics. Some alternative formulations present in the literature are summarized in the next Section.

Let Ω be the region of $\mathcal{E}^N$ occupied by a continuous body. The interaction of a body with the exterior is assumed to consist of two vector-valued measures, a volumic measure $\mu : \overline{\wp}(\Omega) \to \mathbb{R}^N$ called the *distance action*, and a surface measure $Q : \vec{\mathbb{S}}\,(\Omega) \to \mathbb{R}^N$, called the *contact action*.[27] They are subject to two fundamental axioms, the *Euler laws of motion* and the *cut principle* of Euler and Cauchy.[28]

The Euler laws are the balance laws of *linear momentum* and of *angular momentum*. They state that the total action and the total moment exerted on Ω by the exterior are zero[29]

$$\int_{\Omega} d\mu + \int_{\partial^*\Omega} dQ = 0\,, \qquad \int_{\Omega} x \times d\mu + \int_{\partial^*\Omega} x \times dQ = 0\,. \tag{5.1}$$

The cut principle states that the same balance laws hold for every subbody Π of Ω. This hypothesis requires, in particular, that the contact action be defined on all surfaces $\partial^*\Pi$ and not only on the boundary $\partial^*\Omega$, and that

[26] Because f is integrable by assumption, not only T, but also its divergence is integrable. The regularity of T has been successively relaxed to L^p with divergence in L^p (Ziemer 1983; Šilhavý 1985), to L^∞ with divergence measure (Chen & Torres 2005), to L^1 with divergence measure (Degiovanni et al. 1999), and to measures with divergence measure (Chen & Frid 1999).

[27] Inertia forces are included in μ, see (Noll 1963), Sect. 7.

[28] (Truesdell 1991), Sect. III.1.

[29] Here and in the following, with the same symbol x we denote both a point in $\mathcal{E}^N$ and the position vector $x - o$ with respect to an origin o chosen once and for all.

(5.1) holds with Ω replaced by Π. In other words, *the cut principle requires that Q be a Cauchy flux*.

The measures μ and Q are supposed to have a volume density b and an area density $s(\,\cdot\,,\partial^*\Pi)$, respectively. Then the balance laws take the form

$$\begin{aligned}\int_\Omega b(x)\,dV + \int_{\partial^*\Omega} s(x,\partial^*\Omega)\,dA = 0\,,\\ \int_\Omega x\times b(x)\,dV + \int_{\partial^*\Omega} x\times s(x,\partial^*\Omega)\,dA = 0\,.\end{aligned} \tag{5.2}$$

By the cut principle, the same laws hold for every subbody Π.[30] Then, using the arbitrariness of Π, from the first balance law the dependence of $s(x,\,\cdot\,)$ on the normal, the *action-reaction law*

$$s(x,n) = -s(x,-n)\,, \tag{5.3}$$

and the existence of a stress tensor T such that

$$s(x,n) = T(x)\,n\,, \tag{5.4}$$

are deduced using Theorems 4.3 and 4.4 with f replaced by b.[31] As a consequence, the balance equations (5.2) are reduced to the local forms

$$\mathrm{div}T(x) + b(x) = 0\,, \qquad T(x) = T^T(x) \qquad \text{a.e. } x\in\Omega\,. \tag{5.5}$$

These are the local equations of motion, or, in the absence of inertia forces, the *local equilibrium equations* at the internal points of Ω. After introducing a set $\mathcal{V}$ of *virtual displacements* v, equation $(5.5)_1$ multiplied by v and integrated over Π, the Gauss-Green formula (2.8), the relation (5.4), and the symmetry condition $(5.5)_2$ lead to the *equation of virtual power*

$$\int_\Pi b(x)\cdot v(x)\,dV + \int_{\partial^*\Pi} s(x,n)\cdot v(x)\,dA = \int_\Pi T(x)\cdot\nabla^S v(x)\,dV. \qquad \forall v\in\mathcal{V}. \tag{5.6}$$

This equation states the equality of the *external power* of the actions b, s with the *internal power*, given by the product of the *internal force* T by

[30] In fact, writing equation (5.2) with Ω replaced by Π requires some physical assumptions. Namely, using the same volume density b for Ω and for Π requires the assumption that the distance actions between parts of the body are negligible. Also, treating the contact actions at the interior surfaces of the body in the same way as the contact actions at the boundary means to exclude any special structure of the body's surface, such as, for example, the structure of the *material surfaces* studied in (Gurtin & Murdoch 1975).

[31] For the deduction of (5.3) see (Noll 1959).

the *generalized deformation* $\nabla^S v$. This equation has been deduced from the equilibrium equations (5.5). Conversely, if (5.6) is assumed to hold, equations (5.5) follow after replacing $s(x,n)$ by $T(x)n$. Thus, equation (5.6) is an alternative definition of equilibrium. It can be regarded as the weak form of the definition of an *equilibrated system of actions*.

Assume that the external body forces b be known, that surface tractions $s(x)$ be prescribed on a portion $\partial^*_s\Omega$ of $\partial^*\Omega$, and that null displacements be prescribed on the complement $(\partial^*_s\Omega)^c$ of $\partial^*_s\Omega$. Denoting by $\mathcal{V}_o$ the set of all virtual displacements which vanish on $(\partial^*_s\Omega)^c$, from equation (5.6) written for $\Pi = \Omega$ it follows that

$$\int_\Omega b(x)\cdot v(x)\, dV + \int_{\partial^*_s\Omega} s(x)\cdot v(x)\, dA = \int_\Omega T(x)\cdot\nabla^S v(x)\, dV \quad \forall v\in\mathcal{V}_o\,. \tag{5.7}$$

By introducing the constitutive equation of an elastic material[32]

$$T = g(\nabla u)\,, \tag{5.8}$$

the *weak form of the equilibrium problem* for an elastic body is obtained.[33] For non-elastic bodies the formulation is more complicated, since it requires the introduction of additional variables and of the corresponding generalized forces and evolution equations.[34] Non-elastic continua will not be considered in this paper.

6 Alternative approaches

The traditional approach to continuum mechanics illustrated in the previous Section grew over the centuries, starting from the pioneering work of Newton, Euler, and Cauchy. Relatively recent is the realization that the balance laws (5.2) are consequences of a more fundamental physical principle, the indifference, that is, the invariance under changes of observer, of the external power.[35] For the external power given by the left side of

[32]Here and in the following, the constitutive equations are relative to the current deformed configuration of the body, taken as the reference configuration. This is the reason why the kinematical variable is the gradient of the virtual displacement.

[33]See (Ciarlet 1988), Theorem 5.2-1.

[34]see e.g. (Halphen & Nguyen 1975).

[35](Noll 1963). In Lagrangian Mechanics, the deduction of the balance of linear momentum from the translational indifference of the Lagrangian is a well known consequence of Noether's theorem on the correspondence between the indifference properties of a functional and conservation laws, see e.g. (Lanczos 1949), p. 403.

equation (5.6)

$$P_{ext}(\Pi, v) = \int_{\Pi} b(x) \cdot v(x)\, dV + \int_{\partial^* \Pi} s(x, n) \cdot v(x)\, dA\,, \tag{6.1}$$

indifference is expressed by the condition

$$P_{ext}(\Pi, v) \;=\; P_{ext}(\Pi, v + a + W(\,\cdot\,))\,, \tag{6.2}$$

to be satisfied for all $\Pi \in \wp(\Omega)$, for all vectors a and for all skew-symmetric tensors W.[36] By the linear dependence of P_{ext} on v, this condition holds if and only if

$$P_{ext}(\Pi, a) \;=\; 0\,, \qquad P_{ext}(\Pi, W(\,\cdot\,)) = 0\,, \tag{6.3}$$

for all a and for all W, and from these conditions the balance equations (5.2) easily follow.

Recently, alternative approaches appeared in the literature. One of them consists in taking the equation of virtual power (5.6) as a postulate. In this case, the weak formulation (5.7) of the equilibrium problem follows directly from (5.6) written for $\Pi = \Omega$ and $v \in \mathcal{V}_o$, and the balance equations (5.2) follow from the same equation written for $v = a$ and $v = W(\,\cdot\,)$, respectively. This approach has been largely used to construct models of continua with microstructure.[37] Some of its advantages and drawbacks will be discussed later.

Very common is also the *variational approach*, which consists in minimizing an energy functional, whose Euler equation coincides with equation (5.7). For an elastic continuum, the energy is

$$E(v) = \int_{\Omega} w(\nabla v(x))\, dV - \int_{\Omega} b(x) \cdot v(x)\, dV - \int_{\partial\Omega} s(x) \cdot v(x)\, dA\,, \tag{6.4}$$

where w is the *strain energy density*, and $T = dw(\nabla v)/\, d\nabla v$ is the constitutive equation. Thus, in the variational approach the constitutive equation enters from the very beginning. Since it is our intention to keep the equilibrium conditions separate from the constitutive assumptions, the variational approach will not be considered here.

In the approach based on the properties of bounded Cauchy fluxes introduced in Section 4, the pseudobalance equation (4.10) holds. A comparison of its local form (4.18) with $(5.5)_1$ leads to the identification

$$f \;=\; b\,. \tag{6.5}$$

[36] Here a is the rigid translation $a(x) = a$, and W is the skew-symmetric tensor $Wx = w \times x$ associated with the rotation vector w, see Appendix B.

[37] See (Germain 1983*a*; 1983*b*).

That is, the volume density f of Q coincides with the body force. With this identification, the procedure leading to the equilibrium problem becomes identical to the one followed in the traditional approach.

Thus, at a first glance, the difference of the two approaches looks irrelevant. On the contrary, as pointed out in the Introduction, when the external actions involve additional measures μ^α, Q^α, some problems are met in defining additional microscopic balance equations with the traditional approach. As shown in the following Section, the assumption that all Q^α are bounded Cauchy fluxes provides a quite general solution to this difficulty.

7 Non-classical continua

A *continuum with microstructure* is a continuum in which the deformations act on two length scales of different order of magnitude, macroscopic and microscopic.[38] The macrodeformation is described by the displacement vector u, and the microdeformation is described by a finite number of *order parameters* d^α, also called *internal variables* or *state variables*, defined on finite dimensional inner product spaces Y^α. Each order parameter describes a *microstructure*, and each set of order parameters describes a *continuum with microstructure.*

Here the terms *non-classical continua* and *classical continua* will be used to denote continua with and without microstructure, respectively. It is worth mentioning that a self-contained treatment of classical continua may be obtained by restricting the arguments that follow to the case where the order parameters and the internal variables are absent.

Just as the macrodeformation u is associated with a pair (μ, Q) of vector-valued measures describing the macroscopic external actions, to each d^α corresponds a pair (μ^α, Q^α) of Y^α-valued measures, describing the microscopic external actions due to the α-th microstructure. We confine our attention to the case in which all measures μ^α are absolutely continuous with respect to the volume measure, and all Q^α are absolutely continuous with respect to the area measure.[39] In this case each μ^α has a volume density, the *body*

[38] Historically, the first example of a continuum with microstructure is the continuum with couple stresses (Cosserat E. & F. 1909). A short history of successive developments and a broad list of applications can be found in the book (Capriz 1989).

[39] In general, this assumption is too restrictive. Stress concentrations corresponding to singular Cauchy fluxes appear even in some classical problems of linear elasticity. Examples of concentrated contact interactions are discussed in (Podio Guidugli 2004), and examples of stress fields equilibrated with continuous surface tractions at the boundary and exhibiting, at the interior, stress concentrations on singular surfaces or

microforce β^α, each Q^α has a surface density, the *surface microtraction* σ^α, and the external power has the form

$$P_{ext}(\Pi, v, \nu^\alpha) = \int_\Pi (b \cdot v + \beta^\alpha \cdot \nu^\alpha)\, dV + \int_{\partial^*\Pi} (s \cdot v + \sigma^\alpha \cdot \nu^\alpha)\, dA\,, \quad (7.1)$$

where ν^α are virtual variations of the order parameters d^α.[40]

We also assume that Q and all Q^α are bounded Cauchy fluxes, and we extend to continua with microstructure the approach based on bounded Cauchy fluxes, described in Section 4 for classical continua.

For a bounded Cauchy flux, the pseudobalance equation (4.10) holds for Q, and for each Q^α the pseudobalance equation

$$\int_\Pi \phi^\alpha\, dV + \int_{\partial^*\Pi} \sigma^\alpha\, dA \,=\, 0\,, \quad (7.2)$$

holds as well, with ϕ^α the volume density associated with the flux Q^α. By Theorems 4.3 and 4.4, the relations

$$s \,=\, T\, n\,, \qquad \sigma^\alpha \,=\, \Sigma^\alpha\, n\,, \quad (7.3)$$

and the local pseudobalance equations

$$\mathrm{div} T + f = 0\,, \qquad \mathrm{div}\Sigma^\alpha + \phi^\alpha = 0\,, \quad (7.4)$$

follow, where each Σ^α is a linear map on the corresponding Y^α.[41]

It is convenient to decompose the volume densities f, ϕ^α into the sums

$$f \,=\, b - z\,, \qquad \phi^\alpha \,=\, \beta^\alpha - \zeta^\alpha\,, \quad (7.5)$$

where z and ζ^α measure the deviations of the external body forces b, β^α from the densities f and ϕ^α.

Using (7.4), (7.5), and the Gauss-Green formula (2.8), the external power (7.1) tranforms into the *internal power*

$$P_{int}(\Pi, v, \nu^\alpha) = \int_\Pi (z \cdot v + T \cdot \nabla v + \zeta^\alpha \cdot \nu^\alpha + \Sigma^\alpha \cdot \nabla \nu^\alpha)\, dV\,. \quad (7.6)$$

lines are given in (Lucchesi et al 2006, 2009). An appropriate environment for the study of stress concentrations is provided by the stress fields with divergence measure, see (Degiovanni et al. 1999; Chen & Frid 1999; Chen & Torres 2005; Šilhavý 2008).

[40] Summation over repeated superscripts α is understood. For simplicity of notation, from here onwards the argument x is omitted.

[41] For Y^α equal to $\mathbb{R}$, $\mathbb{R}^N$, $\mathbb{R}^{N\times N}$, the values $\Sigma^\alpha(x)$ are vectors, second-order tensors, and third-order tensors, respectively.

This is the sum of four terms, each of which is the scalar product of an *internal force* by the corresponding *generalized deformation*. Therefore, the terms z, ζ^α in (7.5) are characterized as internal forces, and v and ν^α are the corresponding generalized deformations.

The difference between the internal power (7.6) and the internal power (5.6) of a classical continuum is not only the presence of microstructural terms. Indeed, in (7.6) there is the extra term $(z \cdot v)$, and the generalized deformation $\nabla^S v$ is replaced by ∇v. This is due to the indifference requirements. In all examples to be discussed below, the internal power has the translational indifference property[42]

$$P_{int}(\Pi, a, 0) = 0\,, \tag{7.7}$$

which implies

$$z = 0\,, \tag{7.8}$$

and, therefore, leads to the same identification $f = b$ found for the classical continuum. On the contrary, the condition of rotational indifference varies according to the physical nature of the order parameters. Therefore, the symmetry of T found for a classical continuum is not preserved, in general, in non-classical continua. These are the reasons for the differences between (7.6) and (5.6) remarked above.

As a consequence of (7.8), z can be dropped from the list of the internal forces and, consequently, v can be dropped from the list of the generalized deformations. With this modification, the *equation of virtual power for a continuum with microstructure* takes the form

$$\begin{aligned} \int_\Pi (b \cdot v + \beta^\alpha \cdot \nu^\alpha)\, dV + \int_{\partial^* \Pi} (s \cdot v + \sigma^\alpha \cdot \nu^\alpha)\, dA \\ = \int_\Pi (T \cdot \nabla v + \zeta^\alpha \cdot \nu^\alpha + \Sigma^\alpha \cdot \nabla \nu^\alpha)\, dV\,. \end{aligned} \tag{7.9}$$

Just as equation (5.6) for the classical continuum, this equation defines an *equilibrated system of actions* for the continuum with microstructure.

[42] Just like equation (3.7) for classical continua, in the present approach the equation of virtual power is in fact an identity, which holds when all systems of contact actions are strongly balanced Cauchy fluxes. By consequence, the external power is indifferent if and only if the internal power is. In the following, we will systematically impose the indifference of the internal power.

The *constitutive equations* are relations between the internal forces and the generalized deformations

$$\begin{aligned} T &= \hat{T}\,(\nabla v, \nu^\alpha, \nabla\nu^\alpha)\,, \\ \zeta^\alpha &= \hat{\zeta}^\alpha(\nabla v, \nu^\alpha, \nabla\nu^\alpha)\,, \\ \Sigma^\alpha &= \hat{\Sigma}^\alpha(\nabla v, \nu^\alpha, \nabla\nu^\alpha)\,. \end{aligned} \tag{7.10}$$

Substituting into equation (7.9) written for $\Pi = \Omega$ and imposing boundary conditions either to the kinematical variables u, d^α or to the corresponding contact actions s, σ^α, a generalization of equation (5.7) is obtained. This is the *weak form of the equilibrium problem* for an elastic continuum with microstructure.

In models based on generalizations of the traditional approach, the pseudobalance equation $(7.4)_1$ is replaced by the balance equation of linear momentum $(5.5)_1$. Equations identical to $(7.4)_2$, called *microforce balance equations*, are either postulated,[43] or deduced from the equation of virtual power, assumed as a basic postulate.[44] In both cases, the way these equations are introduced is not completely satisfactory. In the first case, it is difficult to attribute to such equations, which are so strongly dependent on the number and physical nature of the order parameters, the same status of the balance equation of linear momentum, which is a general law of mechanics.[45] In the second case, it is not clear how much liberty is allowed in the choice of the internal power, once the external power has been defined. The choice cannot be completely arbitrary, because it must be compatible with the balance laws (5.2). Then there must be a tacit pre-selection of *admissible* forms of the internal power, in contrast with the character of a postulate attributed to the equation of virtual power.

On the contrary, the approach based on bounded Cauchy fluxes provides *the most general* form of the internal power compatible with the assumed external power (7.1), that is, with the chosen order parameters. We say that this choice determines the *structural properties* of a continuum. Together with the indifference requirements, these properties define specific *classes of continua*. Within each class, the constitutive equations (7.10) define *subclasses of materials*. In many models present in the literature, the assumed

[43] See e.g. (Capriz 1989), Sect. 8. It is interesting that the assumption that z is zero while the ζ^α need not be zero, made on page 22 of the book, coincide with our conclusions deduced from the indifference of power.

[44] This is the method proposed by Germain. Examples of models constructed in this way are given in (Gurtin 2003), Footnote 1, and in (Del Piero 2012), Chapter 2.

[45] An alternative is to view the microforce balance equations as constitutive assumptions. To my knowledge, the nature of these equations has never been clearly specified.

form of the internal power is a special case of (7.6), resulting from a mix of structural properties and constitutive assumptions.[46]

A bounded Cauchy flux is skew-symmetric, and this is the property from which the pseudobalance equations (4.10) and (7.2) follow. From the foundational viewpoint, I find this fact very intriguing. Indeed, the balance equations of linear and angular momentum are expressions of Newton's first two laws of motion, and the skew-symmetry of Q is an expression of the third law, or law of action and reaction. For classical continua, as seen in Section 5, the third law is a consequence of the first two. Accordingly, in classical continuum mechanics the third law is not considered as a general principle.[47] For continua with microstructure, the third law gives a precise status to the microforce balance equations, transforming them into pseudobalance equations. In this respect, the third law recovers the role of a general principle.

8 Continua with indifferent microstructure

Here and in the following Sections, we assume that the translational indifference condition (7.7) holds in general. For the rotational condition, consider first the case in which all microstructures are indifferent

$$P_{ext}(\Pi, W(\,\cdot\,), 0) = 0\,. \tag{8.1}$$

Because scalars are invariant under changes of observer, this is the case of all microstructures whose order parameters are scalars. Examples are given by the scalar theories of damage, strain-gradient plasticity, and crystal plasticity.[48]

The result of this condition applied to the internal power (7.6) is the symmetry of T. In this way condition $(5.5)_2$, which in the traditional approach was a consequence of the balance of angular momentum, is recovered. There are no restrictions on the virtual velocities ζ^α and Σ^α. In the constitutive equations (7.10), the only change is that now the values of the constitutive function $\hat{T}$ are symmetric second-order tensors.

[46] In this respect, it is instructive to compare the two models for strain-gradient plasticity of (Gurtin 2003) and (Gurtin 2004), see (Del Piero 2012), Section 2.4.

[47] See e.g. (Truesdell 1991), Sect. 12.

[48] See e.g. (Del Piero 2012), Sect. 2.2.

9 Micropolar continua

Micropolar continua are continua with microstructure whose order parameters d^α are vector fields, called *directors*. They may represent, for example, the orientations of a crystalline lattice or the directions of some crystal defects. Since the directors change their orientation with the body's deformation, the rotational indifference requires the invariance of the internal power under simultaneous rigid rotations of the body and of the directors

$$P_{int}(\Pi,\, W(\,\cdot\,),\, Wd^\alpha(\,\cdot\,)) = 0\,. \tag{9.1}$$

From (7.6) with $z = 0$ it follows that

$$\begin{aligned} 0 = \int_\Pi \big(T \cdot W + \zeta^\alpha \cdot W d^\alpha + \Sigma^\alpha \cdot W \nabla d^\alpha\big)\, dV \\ = \; W \cdot \int_\Pi \big(T + \zeta^\alpha \otimes d^\alpha + \Sigma^\alpha\, \nabla^T d^\alpha\big)\, dV\,, \end{aligned} \tag{9.2}$$

where $\nabla^T d^\alpha$ is the transpose of ∇d^α. This implies the symmetry of the integrand function. Then T is not symmetric in general, and its skew-symmetric part is

$$T^W = -\big(\Sigma^\alpha\, \nabla^T d^\alpha + \zeta^\alpha \otimes d^\alpha\big)^W\,. \tag{9.3}$$

We say that T^W is the *reactive part* of the internal force T, and that the symmetric part T^S is the *active part.* While T^W is a known function of the microstructural internal forces ζ^α and Σ^α, T^S has to be determined by a constitutive equation. The internal power takes the form

$$\begin{aligned} \int_\Pi \big(T^S \cdot \nabla v + \zeta^\alpha \cdot \nu^\alpha - (\Sigma^\alpha\, \nabla^T d^\alpha + \zeta^\alpha \otimes d^\alpha)^W \cdot \nabla v + \Sigma^\alpha \cdot \nabla \nu^\alpha\big)\, dV \\ = \int_\Pi \big(T^S \cdot \nabla^S v + \zeta^\alpha \cdot (\nu^\alpha - \nabla^W v\, d^\alpha) + \Sigma^\alpha \cdot (\nabla \nu^\alpha - \nabla^W v\, \nabla d^\alpha)\big)\, dV. \end{aligned} \tag{9.4}$$

It shows that the generalized deformations corresponding to the internal forces T^S, $\zeta^\alpha, \Sigma^\alpha$ are

$$\nabla^S v\,, \quad \psi^\alpha = \nu^\alpha - \nabla^W v\, d^\alpha, \quad \Psi^\alpha = \nabla \nu^\alpha - \nabla^W v\, \nabla d^\alpha, \tag{9.5}$$

respectively. Note that ψ^α is the relative rotation between the director d^α and the corresponding direction in the deformed body. Thus, the equation of virtual power becomes

$$\begin{aligned} \int_\Pi (b \cdot v + \beta^\alpha \cdot \nu^\alpha)\, dV + \int_{\partial^* \Pi} (s \cdot v + \sigma^\alpha \cdot \nu^\alpha)\, dA \\ = \int_\Pi (T^S \cdot \nabla^S v + \zeta^\alpha \cdot \psi^\alpha + \Sigma^\alpha \cdot \Psi^\alpha)\, dV\,, \end{aligned} \tag{9.6}$$

and the constitutive equations (7.10) take the form

$$\begin{aligned} T^S &= \hat{T}^S(\nabla^S v, \psi^\alpha, \Psi^\alpha)\,, \\ \zeta^\alpha &= \hat{\zeta}^\alpha(\nabla^S v, \psi^\alpha, \Psi^\alpha)\,, \\ \Sigma^\alpha &= \hat{\Sigma}^\alpha(\nabla^S v, \psi^\alpha, \Psi^\alpha)\,. \end{aligned} \tag{9.7}$$

Equation (9.6) defines an *equilibrated system of actions* for a micropolar continuum. When coupled with the constitutive equations (9.7), it provides the *weak form for the equilibrium problem* for a micropolar continuum.

Special micropolar continua are the *Cosserat continua.* In a three-dimensional body, they are characterized by three mutually orthogonal directors d^α, whose virtual velocities are

$$\nu^\alpha = \omega \times d^\alpha, \tag{9.8}$$

with ω a vector field. With this assumption the directors preserve length and mutual orthogonality in all infinitesimal deformations, and ω measures their common rotation. In the expression (7.1) of the external power, we have

$$\beta^\alpha \cdot \nu^\alpha = \beta^\alpha \cdot \omega \times d^\alpha = d^\alpha \times \beta^\alpha \cdot \omega\,, \tag{9.9}$$

and, similarly,

$$\sigma^\alpha \cdot \nu^\alpha = d^\alpha \times \sigma^\alpha \cdot \omega\,. \tag{9.10}$$

By setting

$$c = d^\alpha \times \beta^\alpha\,, \qquad m = d^\alpha \times \sigma^\alpha\,, \tag{9.11}$$

(with α summed), the external power takes the form

$$P_{ext}(\Pi, v, \omega) = \int_\Pi (b \cdot v + c \cdot \omega)\, dV + \int_{\partial^* \Pi} (s \cdot v + m \cdot \omega)\, dA\,. \tag{9.12}$$

Here c and m are the *body couple* and the *surface couple*, respectively. Thus, due to assumption (9.8), the Cosserat continuum is a micropolar continuum with a single vectorial microstructure.

If all σ^α are the densities of bounded Cauchy fluxes, by the definition of the cross product of a vector and a tensor given in Appendix B, from (7.3) we get

$$m = d^\alpha \times \sigma^\alpha = d^\alpha \times (\Sigma^\alpha n) = (d^\alpha \times \Sigma^\alpha)\, n = Mn\,, \tag{9.13}$$

where $M = d^\alpha \times \Sigma^\alpha$ is the *couple stress.* Then using the Gauss-Green formula and setting

$$\zeta \;=\; c + \mathrm{div} M\,. \tag{9.14}$$

the external power transforms into the internal power

$$P_{int}(\Pi, v, \omega) = \int_{\Pi} \big(T \cdot \nabla v + \zeta \cdot \omega + M \cdot \nabla \omega \big)\, dV\,. \tag{9.15}$$

The rotational indifference condition (9.1) now requires that

$$P_{int}(\Pi, a \times (\,\cdot\,),\, a) = 0\,, \tag{9.16}$$

for all constant vectors a. The identity $a \times x = (a \times I)\, x$ and the relation (B.7) in Appendix B imply

$$T(x) \cdot \nabla (a \times x) = T(x) \cdot (a \times I) = 2\, t(x) \cdot a\,, \tag{9.17}$$

with t the vector associated with the skew-symmetric part of T. Then, by (9.15) and (9.16),

$$2\, t + \zeta \;=\; 0\,. \tag{9.18}$$

This determines the skew-symmetric part of T in the case of Cosserat continua.

In the macroscopic deformation, the infinitesimal rotation is represented by the vector associated with the skew-symmetric part of ∇v. By (B.8), this vector is one half of $\operatorname{curl} v$. Then, again from (B.7),

$$\begin{aligned} T \cdot \nabla v + \,\zeta \cdot \omega \;=\; & T^S \cdot \nabla^S v + \,2\, t \cdot \tfrac{1}{2} \operatorname{curl} v + \zeta \cdot \omega \\ =\; & T^S \cdot \nabla^S v + \,\zeta \cdot (\omega - \tfrac{1}{2} \operatorname{curl}\, v)\,. \end{aligned} \tag{9.19}$$

Then the internal power further reduces to

$$P_{int}(\Pi, v, \omega) = \int_{\Pi} \big(T^S \cdot \nabla^S v + \zeta \cdot (\omega - \tfrac{1}{2} \operatorname{curl} v) + M \cdot \nabla \omega \big)\, dV\,, \tag{9.20}$$

and the constitutive equations (9.7) take the form

$$\begin{aligned} T^S &= \hat{T}^S (\nabla^S v, \omega - \tfrac{1}{2} \operatorname{curl} v, \nabla \omega)\,, \\ \zeta &= \;\hat{\zeta}\; (\nabla^S v, \omega - \tfrac{1}{2} \operatorname{curl} v, \nabla \omega)\,, \\ M &= \hat{M} (\nabla^S v, \omega - \tfrac{1}{2} \operatorname{curl} v, \nabla \omega)\,. \end{aligned} \tag{9.21}$$

Comparison with the constitutive equations (9.7) shows that the relative rotation $(\omega - \tfrac{1}{2} \operatorname{curl} v)$ corresponds to the relative rotation ψ^{α} of the general micropolar continuum.

Again, the equation of virtual power obtained by equating the powers (9.12) and (9.20) defines an equilibrated system of actions for a Cosserat continuum, and this same equation coupled with the constitutive equations (9.21) provides the corresponding weak form of the equilibrium problem.

10 Second-gradient continua

A second-gradient continuum is a continuum with microstructure whose unique order parameter is the displacement gradient ∇u.[49] The corresponding virtual velocity is ∇v, and the external power has the form

$$P_{ext}(\Pi, v) = \int_{\Pi} (b \cdot v + B \cdot \nabla v)\, dV + \int_{\partial^* \Pi} (s \cdot v + S \cdot \nabla v)\, dA\,. \tag{10.1}$$

If the second-order tensor field S is the surface density of a bounded Cauchy flux, the tensorial version[50] of Theorems 4.3 and 4.4 ensures the existence of a third-order tensor field $\mathbb{T}$ such that

$$\begin{aligned} S &= \mathbb{T} n\,, & \mathrm{div}\mathbb{T} + \Phi &= 0\,, \\ S_{ij} &= \mathbb{T}_{ijk} n_k\,, & \mathbb{T}_{ijk,k} + \Phi_{ij} &= 0\,, \end{aligned} \tag{10.2}$$

where Φ is the volume density associated with the Cauchy flux. The internal power, given by the right-hand side of (7.9), now has the form

$$P_{int}(\Pi, v) = \int_{\Pi} \big((T + Z) \cdot \nabla v + \mathbb{T} \cdot \nabla\nabla v\big)\, dV, \tag{10.3}$$

where $Z = B - \Phi$ is the gap between the external body microforce B and Φ. The rotational indifference is expressed by condition $(6.3)_2$, which now implies the symmetry of $(T + Z)$

$$(T + Z)^W = 0\,. \tag{10.4}$$

Then the generalized deformation associated with the internal force $(T+Z)$ is $\nabla^S v$ instead of ∇v. Moreover, due to the symmetry of $\nabla\nabla v$ with respect to the last two subscripts, only the part of $\mathbb{T}$ symmetric with respect to the last two subscripts, here denoted by $\mathbb{T}^{\times S}$, contributes to the power. Therefore, for a second-gradient continuum the internal forces are the symmetric tensors $(T + Z)$ and $\mathbb{T}^{\times S}$, and $\nabla^S v$ and $\nabla\nabla v$ are the corresponding generalized deformations. For an elastic material, the constitutive equations are

$$\begin{aligned} T + Z &= \hat{F}(\nabla^S v, \nabla\nabla v)\,, \\ \mathbb{T}^{\times S} &= \hat{\mathbb{T}}(\nabla^S v, \nabla\nabla v)\,, \end{aligned} \tag{10.5}$$

[49] (Toupin 1962; Mindlin 1964; Germain 1973a). See also (Forte & Vianello 1988; Noll & Virga 1990; Dell'Isola & Seppecher 1997; Podio Guidugli & Vianello 2010).

[50] See e.g. the Appendix of (Del Piero 2009).

with $\hat{F}$ symmetric, and with $\hat{\mathbb{T}}$ symmetric with respect to the last two subscripts.[51]

The name of *second-order continuum* is due to presence of $\nabla\nabla v$ among the generalized deformations. This presence causes some complication in the formulation of the boundary conditions. Indeed, the displacement gradient at the boundary has a normal and a tangential component, and the tangential component is determined by the values of v at the boundary. Then, boundary conditions of place can be prescribed only to v and to the normal component of ∇v.

The form of the boundary conditions is a part of the *strong formulation* of the equilibrium problem. For a three-dimensional body Ω, at the boundary $\partial^*\Omega$ take an orthonormal local reference frame (e^α, e^n), where e^α, $\alpha \in \{1,2\}$, are tangent vectors, and e^n is the exterior normal n. After decomposing the product $S \cdot \nabla v$ into the sum of a normal and a tangential part, the Gauss-Green formula applied to $\partial^*\Omega$ yields

$$\begin{aligned}
&\int_{\partial^*\Omega} S \cdot \nabla v \, dA = \int_{\partial^*\Omega} (S_{in} v_{i,n} + S_{i\alpha} v_{i,\alpha}) \, dA \\
&= \int_{\partial^*\Omega} (S_{in} v_{i,n} - S_{i\alpha,\alpha} v_i) \, dA = \int_{\partial^*\Omega} (Sn \cdot \nabla_n v - \mathrm{div}_\alpha S \cdot v) \, dA \, .
\end{aligned} \tag{10.6}$$

In a similar way, a double application of the Gauss-Green formula provides a well-known transformation of the second-gradient term[52]

$$\begin{aligned}
&\int_\Omega \mathbb{T} \cdot \nabla\nabla v \, dV = \int_\Omega \mathbb{T}^{\times S}_{ijk} v_{i,jk} \, dV \\
&= -\int_\Omega \mathbb{T}^{\times S}_{ijk,k} v_{i,j} \, dV + \int_{\partial^*\Omega} \mathbb{T}^{\times S}_{ijn} \, v_{i,j} \, dA \\
&\quad = \int_\Omega \mathbb{T}^{\times S}_{ijk,kj} v_i \, dV + \int_{\partial^*\Omega} (\mathbb{T}^{\times S}_{inn} v_{i,n} + \mathbb{T}^{\times S}_{i\alpha k} n_k v_{i,\alpha}) - \mathbb{T}^{\times S}_{ink,k} v_i) \, dA \\
&\quad = \int_\Omega \mathbb{T}^{\times S}_{ijk,kj} v_i \, dV + \int_{\partial^*\Omega} \big(\mathbb{T}^{\times S}_{inn} v_{i,n} - (\mathbb{T}^{\times S}_{i\alpha n,\alpha} + \mathbb{T}^{\times S}_{ink,k}) \, v_i\big) \, dA \, .
\end{aligned} \tag{10.7}$$

By the symmetry of $\mathbb{T}^{\times S}$,

$$\begin{aligned}
\mathbb{T}^{\times S}_{i\alpha n,\alpha} &= \mathbb{T}^{\times S}_{in\alpha,\alpha} = (\mathrm{div}_\alpha(\mathbb{T}^{\times S} n))_i \, , \\
\mathbb{T}^{\times S}_{ink,k} &= \mathbb{T}^{\times S}_{in\alpha,\alpha} + \mathbb{T}^{\times S}_{inn,n} = (\mathrm{div}_\alpha(\mathbb{T}^{\times S} n))_i + (\nabla_n(\mathbb{T}^{\times S} nn))_i \, .
\end{aligned}$$

[51] In particular, the constitutive equations are independent of $\nabla^W v$. See (Grioli 1960), eq. 19, and (Toupin 1962), eq. 5.1-5.3.

[52] (Toupin 1962), (Mindlin 1964).

Therefore,

$$\begin{aligned}\int_\Omega \mathbb{T}\cdot\nabla\nabla v\,dV = \int_\Omega \mathrm{divdiv}\mathbb{T}^{\times S}\cdot v\,dV + \int_{\partial^*\Omega} \mathbb{T}^{\times S} nn\cdot\nabla_n v\,dA \\ -\int_{\partial^*\Omega}\big(2\,\mathrm{div}_\alpha(\mathbb{T}^{\times S} n)+\nabla_n(\mathbb{T}^{\times S} nn)\big)\ \cdot v\,dA\,.\end{aligned} \tag{10.8}$$

After the transformation

$$\int_\Omega B\cdot\nabla v\,dV = -\int_\Omega \mathrm{div}B\cdot v\,dV + \int_{\partial^*\Omega} Bn\cdot v\,dA\,, \tag{10.9}$$

and a similar transformation for $(T+Z)$, the equation of virtual power eventually takes the form

$$\begin{aligned}&\int_\Omega (b-\mathrm{div}B)\cdot v\,dV + \int_{\partial^*\Omega}\big(Sn\cdot\nabla_n v + (Bn+s-\mathrm{div}_\alpha S)\cdot v\big)\,dA \\ =&\int_\Omega \big(\mathrm{divdiv}\mathbb{T}^{\times S}-\mathrm{div}(T+Z)\big)\cdot v\,dV + \int_{\partial^*\Omega}\mathbb{T}^{\times S} nn\cdot\nabla_n v\,dA \\ &+\int_{\partial^*\Omega}\big((T+Z)\,n - 2\,\mathrm{div}_\alpha(\mathbb{T}^{\times S} n) - \nabla_n(\mathbb{T}^{\times S} nn)\big)\ \cdot v\,dA\,.\end{aligned} \tag{10.10}$$

From the arbitrariness of v, the equilibrium equation

$$\mathrm{div}\,\mathrm{div}\mathbb{T}^{\times S} - \mathrm{div}(T+Z)\ =\ b-\mathrm{div}B\,, \tag{10.11}$$

at the interior points, and the conditions

$$\begin{aligned}((\mathbb{T}^{\times S} n)\,n - Sn)\cdot\nabla_n v = 0\,,\\ \big((T+Z)\,n - 2\,\mathrm{div}_\alpha(\mathbb{T}^{\times S} n) - \nabla_n((\mathbb{T}^{\times S} n)\,n) - Bn - s + \mathrm{div}_\alpha S\big)\cdot v = 0\,,\end{aligned} \tag{10.12}$$

at the boundary, are deduced. The latter provide the desired forms of the boundary conditions of traction

$$\begin{aligned}(\mathbb{T}^{\times S} n)\,n\ =\ Sn\,,\\ (T+Z)\,n - 2\,\mathrm{div}_\alpha(\mathbb{T}^{\times S} n) - \nabla_n((\mathbb{T}^{\times S} n)\,n)\ =\ Bn + s - \mathrm{div}_\alpha S\,,\end{aligned} \tag{10.13}$$

on the portion of the boundary on which the values of the contact actions S and s are prescribed.[53]

[53] These are the equations in (Mindlin 1964), in the improved version of (Bleustein 1967), plus the simplification (10.8) due to the symmetry of $\mathbb{T}^{\times S}$. An interesting interpretation of the boundary conditions in terms of *ortho-fibers* is given in (Froiio et al. 2010).

The operator (div_α), obtained in (10.6) and (10.7) when applying the Gauss-Green formula to $\partial^*\Omega$, must be interpreted in the distributional sense. Then, if the boundary $\partial^*\Omega$ has an *edge line*, that is, a line at which the normal is discontinuous, the terms $\mathrm{div}_\alpha S$ in (10.6) and $\mathrm{div}_\alpha(\mathbb{T}^{\times S}n)$ in (10.8) include singularities, called *edge forces*, represented by forces per unit length applied to the edge line.[54]

In (10.6), the power of the *hypertractions* S is represented as the sum of the powers of two ordinary tractions, (Sn) and ($\mathrm{div}_\alpha S$), and the latter has singularities at points at which the normal is discontinuous. Some authors believe that the presence of singularities requires a reformulation of theorems 4.3 and 4.4. This does not seem to be the case, as long as the field S itself is not singular, that is, as long as the Cauchy flux Q^α is absolutely continuous with respect to the area measure. Indeed, in this case the tensorial versions of Theorems 4.3 and 4.4 apply. The singularities are only apparent, because they originate from the representation of the power in a discontinuous local basis.[55]

On the contrary, more regular regions and generalized versions of Noll's and Cauchy's theorems become necessary when the external actions involve singular measures. In spite of some valuable progress,[56] the construction of a comprehensive, self-consistent theory of higher-order continua in the presence of singular measures is still an open problem.

[54] Similarly, in a third-gradient continuum, concentrated forces, called *wedge forces*, appear at vertices, and higher-order terms appear in higher-gradient continua. See (Noll & Virga 1988), (Di Carlo & Tatone 2001), (Podio Guidugli & Vianello 2010), (Dell'Isola et al. 2011). A characterization of the power of a n^{th}-gradient continuum based on the concept of *diffused subbody* was proposed in (Degiovanni et al. 2007).

[55] As stated in (Noll & Virga 1990), "edge interactions should not be confused with external actions concentrated along curves". The problems caused by writing the power equation in terms of tangential and normal components have a long history. Perhaps, they were met for the first time in the theory of the bending of plates. The determination of the boundary conditions for this problem kept scientists of the calibre of Poisson, Lagrange, Kirchhoff, and Kelvin, busy for a good part of the 19-th century. For a history of the problem see (Timoshenko 1953).

[56] The problem of the regularity of regions was posed in (Noll & Virga 1990). *Sets with curvature measure* have been introduced by (Degiovanni et al. 2006). For extensions of theorems 4.3 and 4.4 in the presence of singular measures see (Dell'Isola & Seppecher 1997; Marzocchi & Musesti 2003; Degiovanni et al. 2006; Dell'Isola et al. 2011).

11 Continua with latent microstructure

A continuum with latent microstructure is a continuum subject to internal constraints relating the order parameters to the macroscopic deformation.[57] By effect of the constraint, the internal forces decompose into the sum of an *active part* and of a *reactive part*. The latter does not appear in the constitutive equations and in the equilibrium equations.

An example is the second-gradient continuum described in the preceding Section. It has a single tensorial order parameter, whose virtual velocity $\mathcal{V}^\alpha$ is subject to the constraint

$$\mathcal{V}^\alpha = \nabla v\,. \tag{11.1}$$

By (10.3), the internal forces are $(T+Z)$ and $\mathbb{T}$, and the active parts are $(T+Z)^S$ and $\mathbb{T}^{\times S}$.

A second example is given by the *Cauchy-Born hypothesis*, by which the orientations of the directors d^α are forced to follow the macroscopic deformation. That is, their variations ν^α must satisfy the the internal constraint[58]

$$\nu^\alpha = \nabla v\, d^\alpha, \tag{11.2}$$

In this case, the external power (7.1) takes the form

$$P_{ext}(\Pi, v) = \int_\Pi \big(b \cdot v + (\beta^\alpha \otimes d^\alpha) \cdot \nabla v\big)\, dV + \int_{\partial^*\Pi} \big(s \cdot v + (\sigma^\alpha \otimes d^\alpha) \cdot \nabla v\big)\, dA\,,$$

(with α summed). This is a special case of (10.1), with

$$B = \beta^\alpha \otimes d^\alpha\,, \qquad S = \sigma^\alpha \otimes d^\alpha\,. \tag{11.3}$$

The internal power is

$$P_{int}(\Pi, v) = \int_\Pi \big((T + \Sigma^\alpha\, \nabla^T d^\alpha + \zeta^\alpha \otimes d^\alpha) \cdot \nabla v + (\Sigma^\alpha \otimes d^\alpha) \cdot \nabla\nabla v\big)\, dV, \tag{11.4}$$

with ζ^α as in $(7.5)_2$. This is a special case of (10.3), with

$$Z = \Sigma^\alpha \nabla^T d^\alpha + \zeta^\alpha \otimes d^\alpha\,, \qquad \mathbb{T} = \Sigma^\alpha \otimes d^\alpha. \tag{11.5}$$

Thus, a micropolar continuum obeying the Cauchy-Born hypothesis is a second-gradient continuum with particular forms for B, S, Z, and $\mathbb{T}$.

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[57] (Capriz 1985, 1989).

[58] If the directors d^α form a basis for the underlying space and if d_α is the dual basis, this is a special case of (11.1), with $\mathcal{V}^\alpha = \nu^\alpha \otimes d_\alpha$.

Another special case is the *Cosserat continuum with constrained rotation.*[59] For this continuum, the virtual velocities ν^α are subject to the internal constraint

$$\nu^\alpha \;=\; \nabla^W v\, d^\alpha. \tag{11.6}$$

By (B.4) and (B.8), this constraint can be put in the equivalent form $\nu^\alpha = \frac{1}{2}\,\mathrm{curl}\, v \times d^\alpha$. Comparing with (9.8), we see that the internal constraint reduces to

$$\omega \;=\; \tfrac{1}{2}\,\mathrm{curl}\, v\,. \tag{11.7}$$

That is, the rotations ω of all directors coincide with the rotation $\frac{1}{2}\,\mathrm{curl}\, v$ in the macroscopic deformation. The external power (9.12) reduces to

$$P_{ext}(\Pi, v) = \int_\Pi (b\cdot v + \tfrac{1}{2}\, c\cdot \mathrm{curl}\, v)\, dV + \int_{\partial^*\Pi} (s\cdot v + \tfrac{1}{2}\, m\cdot \mathrm{curl}\, v)\, dA\,, \tag{11.8}$$

and the internal power (9.20) reduces to

$$P_{int}(\Pi, v) \;=\; \int_\Pi (T^S\cdot \nabla^S v + \tfrac{1}{2}\, M\cdot \nabla\mathrm{curl}\, v)\, dV\,. \tag{11.9}$$

Therefore, the generalized deformations corresponding to the internal forces T^S and M are $\nabla^S v$ and $\frac{1}{2}\,\nabla\mathrm{curl}\, v$, respectively. The constitutive equations (9.21) become

$$\begin{aligned} T^S &= \hat{T}^S(\nabla^S v, \tfrac{1}{2}\,\nabla\mathrm{curl}\, v)\,, \\ M &= \hat{M}\,(\nabla^S v, \tfrac{1}{2}\,\nabla\mathrm{curl}\, v)\,. \end{aligned} \tag{11.10}$$

This is still a second-order continuum. Indeed, by (B.8),

$$\tfrac{1}{2}\, M\cdot \mathrm{curl}\, v = \tfrac{1}{2}\, M_{ij} e_{ihk} v_{k,hj} = \mathbb{T}\cdot \nabla\nabla v\,, \tag{11.11}$$

with $\mathbb{T}$ the third-order tensor[60]

$$\mathbb{T}_{khj} = \tfrac{1}{2}\, M_{ij} e_{ihk}\,. \tag{11.12}$$

For the internal forces T, ζ, M in (9.15), the active parts are $T^S, 0, M$, and the reactive parts are T^W, ζ, 0, respectively. The latter do not appear in equations (11.9) and (11.10).

A Appendix. Proof of Theorem 4.3

The proof is divided into four steps.

[59](Toupin 1964).

[60] Note that $\mathbb{T}$ is skew-symmetric with respect to the first two subscripts. Then $\mathbb{T}$ has nine independent components, as many as the couple stress tensor M.

Step 1. For simplicity, write B_r in place of $B_r(x)$. Consider the definition (4.2) for the set $\Pi \cap B_r$

$$F(\Pi \cap B_r) + Q(\partial^*(\Pi \cap B_r)) = 0\,.$$

By Proposition 3.2, the set $\partial^*(\Pi \cap B_r)$ admits the decomposition (3.8)

$$\partial^*(\Pi \cap B_r) = (\Pi \cap \partial^* B_r)^\star \curlyvee (\partial^* \Pi \cap B_r)^\star\,.$$

Then, by the the additivity property (3.10),

$$Q(\partial^*(\Pi \cap B_r)) = Q((\Pi \cap \partial^* B_r)^\star) + Q((\partial^* \Pi \cap B_r)^\star)\,,$$

and the pseudobalance equation reduces to

$$F(\Pi \cap B_r) + Q((\Pi \cap \partial^* B_r)^\star) + Q((\partial^* \Pi \cap B_r)^\star) \;=\; 0\,. \tag{A.1}$$

Step 2. Let us prove that

$$\lim_{r\to 0} \frac{F(\Pi \cap B_r)}{A(\partial^* B_r)} \;=\; 0\,. \tag{A.2}$$

By the property (4.9) of the bounded Cauchy fluxes, there is a positive function $h \in L^1(\Omega, \mathbb{R})$ such that

$$|F(\Pi \cap B_r))| \;=\; |Q(\partial^*(\Pi \cap B_r))| \;\le\; \int_{\Pi \cap B_r} h(x)\, dV\,.$$

Therefore,

$$\lim_{r\to 0} \Big|\frac{F(\Pi \cap B_r)}{A(\partial^* B_r)}\Big| \;\le\; \lim_{r\to 0} \frac{\int_{\Pi \cap B_r} h(x)\, dV}{V(\Pi \cap B_r)}\; \frac{V(\Pi \cap B_r)}{A(\partial^* B_r)}\;.$$

The first term on the right converges to a finite value V-almost everywhere,[61] and the second term converges to zero, because the volume of B_r goes to zero faster than the area. Then, (A.2) follows.

Step 3. Let us prove that

$$\lim_{r\to 0} \frac{Q((\partial^* B_r \cap \Pi)^\star)}{A(\partial^* B_r)} \;=\; \lim_{r\to 0} \frac{Q((\partial^* B_r \cap H)^\star)}{A(\partial^* B_r)}\;, \tag{A.3}$$

[61] By the Lebesgue-Besicovitch differentiation theorem, see e.g. (Evans & Gariepy 1992).

where $H = H(x, n)$ is the half-space with x as a boundary point and with exterior normal n. From the identity

$$\Pi = (\Pi \cap H) \vee (\Pi \cap H^c)$$

it follows that

$$\begin{aligned}(\partial^* B_r \cap \Pi)^\star &= (\partial^* B_r \cap (\Pi \cap H))^\star \curlyvee (\partial^* B_r \cap (\Pi \cap H^c))^\star \,, \\ (\partial^* B_r \cap H)^\star &= (\partial^* B_r \cap (H \cap \Pi))^\star \curlyvee (\partial^* B_r \cap (H \cap \Pi^c))^\star \,.\end{aligned}$$

Then it is sufficient to prove that

$$\lim_{r\to 0} \frac{Q((\partial^* B_r \cap \Pi \cap H^c)^\star)}{A(\partial^* B_r)} = 0\,, \quad \lim_{r\to 0} \frac{Q((\partial^* B_r \cap H \cap \Pi^c)^\star)}{A(\partial^* B_r)} = 0\,. \tag{A.4}$$

Since $H^c = H(x, -n)$, from the definition of the measure-theoretic normal we have

$$\lim_{r\to 0} \frac{V(B_r \cap \Pi \cap H^c)}{V(B_r)} = 0\,. \tag{A.5}$$

On the other hand, by the coarea formula[62]

$$V(B_r \cap \Pi \cap H^c) = \int_0^r A(\partial^* B_\eta \cap \Pi \cap H^c)\, d\eta\,,$$

that is, the map $r \mapsto V(B_r \cap \Pi \cap H^c)$ is differentiable, and

$$\frac{d}{dr} V(B_r \cap \Pi \cap H^c) = A(\partial^* B_r \cap \Pi \cap H^c) \tag{A.6}$$

for almost every r.[63] Since $V(B_r) = O(r^N)$, from (A.5) it follows that $V(B_r \cap \Pi \cap H^c) = o(r^N)$, and from (A.6) it follows that $A(\partial^* B_r \cap \Pi \cap H^c) = o(r^{N-1})$. Because $A(\partial^* B_r) = O(r^{N-1})$, we conclude that

$$\lim_{r\to 0} \frac{A(\partial^* B_r \cap \Pi \cap H^c)}{A(\partial^* B_r)} = 0\,. \tag{A.7}$$

In the identity

$$\begin{aligned}&\lim_{r\to 0} \frac{Q((\partial^* B_r \cap \Pi \cap H^c)^\star)}{A(\partial^* B_r)} \\ &\qquad = \lim_{r\to 0} \frac{Q((\partial^* B_r \cap \Pi \cap H^c)^\star)}{A((\partial^* B_r \cap \Pi \cap H^c)^\star)} \lim_{r\to 0} \frac{A((\partial^* B_r \cap \Pi \cap H^c)^\star)}{A(\partial^* B_r)}\,,\end{aligned}$$

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[62](Ziemer 1983), Eqn. (8).

[63]Again by the Lebesgue-Besicovitch theorem. The same holds for the identity next to (A.7).

by the absolute continuity of Q, the first limit on the right exists and is finite for A-almost every x in Ω. The second limit is zero by (A.7), because a subsurface differs from the corresponding normalized surface at most by a set of area zero. Then $(\mathrm{A}.4)_1$ follows. Equation $(A.4)_2$ is proved in the same way.

Step 4. From (A.1), (A.2), and (A.3) it follows that

$$\lim_{r\to 0} \frac{Q((\partial^*\Pi \cap B_r)^\star)}{A(\partial^* B_r)} = \lim_{r\to 0} \frac{Q((\partial^* H \cap B_r)^\star)}{A(\partial^* B_r)} \,. \tag{A.8}$$

If the first limit exists for some Π, it also exists for H and, by consequence, it exists for any other Π' with normal n at x. On the other hand,

$$\lim_{r\to 0} \frac{Q((\partial^*\Pi \cap B_r)^\star)}{A(\partial^* B_r)} = \lim_{r\to 0} \frac{Q((\partial^*\Pi \cap B_r)^\star)}{A((\partial^*\Pi \cap B_r)^\star)} \lim_{r\to 0} \frac{A((\partial^*\Pi \cap B_r)^\star)}{A(\partial^* B_r)} \,.$$

At A-almost every x on $\partial^*\Pi$, the first limit on the right is equal to $s(x, \partial^*\Pi)$, and the second limit is equal to one.[64] By (A.8), the same conclusion holds with Π replaced by H. This proves that

$$s(x, \partial^*\Pi) = s(x, \partial^* H) \tag{A.9}$$

at all points $x \in \partial^*\Pi$ at which the limit $s(x, \partial^*\Pi)$ exists, that is, A-almost everywhere on $\partial^*\Pi$.

If x is one of such points and if Π' is another surface such that $x \in \partial^*\Pi'$ and n is the normal at x, in the same way as above it can be proved that

$$s(x, \partial^*\Pi') = s(x, \partial^* H) \,. \tag{A.10}$$

Then it is possible to denote by $s(x, n)$ the common value of all $s(x, \partial^*\Pi')$, and (4.13) follows from (A.9) and (A.10). □

B Appendix. The cross product of a vector by a tensor

The cross product of a vector w by a second-order tensor A is the second-order tensor $(w \times A)$ such that[65]

$$(w \times A)\, v = w \times (Av) \tag{B.1}$$

[64] (Ziemer 1983), Eqn. (5).

[65] From (Antman & Osborn 1979), modified.

for all vectors v. In components,

$$(w \times A)_{ij} \;=\; e_{ihk} w_h A_{kj} \,. \tag{B.2}$$

In particular, $w \times I$ is the second-order tensor such that

$$(w \times I)\, v \;=\; w \times v \tag{B.3}$$

for all v. By definition, $(w \times I)$ is the skew-symmetric tensor associated with w. In components, for $W = w \times I$ the relations

$$W_{ij} = e_{ikj} w_k \,, \qquad w_i = \tfrac{1}{2} e_{jik} W_{jk} \tag{B.4}$$

hold. In particular, let V and W be the skew-symmetric tensors associated with v and w, respectively. Then, by (B.1),

$$2w \cdot v = e_{jik} W_{jk} v_i = -(v \times W)_{ii} = -I \cdot (v \times W)\,, \tag{B.5}$$

and, by (B.1) and (B.3),

$$v \times W = (v \times I)\, W = VW \,. \tag{B.6}$$

Then, by the skew-symmetry of V and W,

$$2w \cdot v = -I \cdot VW = V \cdot W \,. \tag{B.7}$$

Finally, consider the curl of a vector field v

$$(\operatorname{curl} v)_h \;=\; e_{hsr} v_{r,s} \,.$$

By (B.2),

$$(\operatorname{curl} v \times I)_{ij} \;=\; e_{ihk} e_{hsr} v_{r,s} \delta_{kj} \;=\; v_{i,j} - v_{j,i} \;=\; 2\,(\nabla^W v)_{ij} \,, \tag{B.8}$$

that is, $2\nabla^W v$ is the skew-symmetric tensor associated with $\operatorname{curl} v$.

Bibliography

[Ambrosio et al. 2000] L. Ambrosio, N. Fusco, D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Clarendon Press, Oxford 2000

[Antman & Osborn 1979] S.S. Antman, J.E. Osborn, The principle of virtual work and integral laws of motion, Arch. Ration. Mech. Anal. 69: 231-262 (1979)

[Bleustein 1967] J.L. Bleustein, A note on the boundary conditions of Toupin's strain-gradient theory, Int. J. Solids Structures 3: 1053-1057 (1967)

[Capriz 1985] G. Capriz, Continua with latent microstructure, Arch. Ration. Mech. Anal. 90: 43-56 (1985)

[Capriz 1989] G. Capriz, *Continua with Microstructure*, Springer, Berlin 1989

[Cauchy 1823] A.L. Cauchy, Recherches sur l' équilibre et le mouvement intérieur des corps solides ou fluides, élastiques ou non élastiques, Bull. Soc. Philomatique, 9-13 (1823). Reprinted in: *Oeuvres* 2: 300-304, Gauthier-Villars, Paris 1889

[Chen & Frid 1999] G-Q. Chen, H. Frid, Divergence measure fields and hyperbolic conservation laws, Arch. Ration. Mech. Anal. 147: 89-118 (1999)

[Chen & Torres 2005] G-Q. Chen, M. Torres, Divergence measure fields, sets of finite perimeter, and conservation laws, Arch. Ration. Mech. Anal. 175: 245-267 (2005)

[Ciarlet 1988] P.G. Ciarlet, *Mathematical Elasticity. Volume 1: Three-Dimensional Elasticity*, North-Holland, Amsterdam 1988

[Cosserat E. & F. 1909] E. and F. Cosserat. *Théorie des corps déformables*, Hermann, Paris 1909

[De Giorgi 1954] E. De Giorgi, Su una teoria generale della misura $(r-1)$-dimensionale in uno spazio ad r dimensioni, Annali di Matematica Pura ed Appl. 36: 191-213 (1954)

[Degiovanni et al. 1999] M. Degiovanni, A. Marzocchi, A. Musesti, Cauchy fluxes associated with tensor fields having divergence measure, Arch. Ration. Mech. Anal. 147: 197-223 (1999)

[Degiovanni et al. 2006] M. Degiovanni, A. Marzocchi, A. Musesti, Edge-force densities and second-order powers, Annali di Matematica 185: 81-103 (2006)

[Degiovanni et al. 2007] M. Degiovanni, A. Marzocchi, A. Musesti, Virtual powers on diffused subbodies and normal traces of tensor-valued measures, in: M. Šilhavý ed., *Mathematical Models of Bodies with Complicated Bulk and Boundary Behavior*, Quaderni di Matematica 20: 21-53 (2007)

[Dell'Isola & Seppecher 1997] F. Dell'Isola, P. Seppecher, Edge contact forces and quasi-balanced power, Meccanica 32: 33-52 (1997)

[Dell'Isola et al. 2011] F. Dell'Isola, P. Seppecher, A.Madeo, Beyond Euler-Cauchy continua. The structure of contact actions in N-th gradient generalized continua: a generalization of the Cauchy tetrahedron argument, CISM Courses and Lectures 535: 17-106 (2011)

[Del Piero 2003] G. Del Piero, A class of fit regions and a universe of shapes for continuum mechanics, J. of Elasticity 70: 175-195 (2003)

[Del Piero 2009] G. Del Piero, On the method of virtual power in continuum mechanics, J. Mech. Mater. Struct. 4: 281-292 (2009)

[Del Piero 2012] G. Del Piero, On the method of virtual power in the mechanics of non-classical continua, Lecture Notes to the CISM Course *Multiscale Modelling of Complex Materials*, Udine 2012. To appear

[Di Carlo & Tatone 2001] A. Di Carlo, A. Tatone, (Iper-)tensioni & equipotenza, Proc. 15-th AIMETA Congress of Theoretical and Applied Mechanics, Taormina 2001

[Evans & Gariepy 1992] L.C. Evans, R.F. Gariepy, *Measure Theory and Fine Properties of Functions*, CRC Press, Boca Raton 1992

[Federer 1958] H. Federer, A note on the Gauss-Green theorem, Proc. AMS 9: 447-451 (1958)

[Federer 1969] H. Federer, *Geometric Measure Theory*, Springer-Verlag, New York 1969

[Forte & Vianello 1988] S. Forte, M. Vianello, On surface stresses and edge forces, Rend: Mat. Appl. VII/8: 409-426 (1988)

[Fosdick & Virga 1989] R.L. Fosdick, E.G. Virga, A variational proof of the stress theorem of Cauchy, Arch. Ration. Mech. Anal. 105: 95-103 (1989)

[Froiio et al. 2010] F. Froiio, A. Zervos, I. Vardoulakis, On natural boundary conditions in linear 2-nd grade elasticity, in: G.A. Maugin, A.V. Metrikine eds., *Mechanics of Generalized Continua. One Hundred Years After the Cosserats*: 211-222, Springer 2010

[Germain 1973a] P. Germain, La méthode des puissances virtuelles en mécanique des milieux continus. Première partie: théorie du second gradient, J. de Mécanique 12: 235-274 (1973)

[Germain 1973b] P. Germain, The method of virtual power in continuum mechanics. Part 2: microstructure, SIAM J. Appl. Math. 25: 556-575 (1973)

[Grioli 1960] G. Grioli, Elasticità asimmetrica, Ann. Mat. Pura Appl. 50: 389-417 (1960)

[Gurtin 2003] M.E. Gurtin, On a framework for small-deformation viscoplasticity: free energies, microforces, strain gradients, Int. J. of Plasticity 19: 47-90 (2003)

[Gurtin 2004] M.E. Gurtin, A gradient theory of small-deformation isotropic plasticity that accounts for the Burgers vector and for dissipation due to plastic spin, J. Mech. Phys. Solids 52: 2545-2568 (2004)

[Gurtin & Martins 1976] M.E. Gurtin, L.C. Martins, Cauchy's theorem in classical physics, Arch. Ration. Mech. Anal. 60: 305-324 (1976)

[Gurtin et al. 1968] M.E. Gurtin, V.J. Mizel, W.O. Williams, A note on Cauchy's stress theorem, J. Math. Anal. Appls. 22: 398-401 (1968)

[Gurtin & Murdoch 1975] M.E. Gurtin, A.I. Murdoch, A continuum theory of elastic material surfaces, Arch. Ration. Mech. Anal. 57: 291-323 (1975)

[Gurtin et al. 1986] M.E. Gurtin, W.O. Williams, W.P. Ziemer, Geometric measure theory and the axioms of continuum thermodynamics, Arch. Ration. Mech. Anal. 92: 1-22 (1986)

[Halphen & Nguyen 1975] B. Halphen, Q.S. Nguyen, Sur les matériaux standards généralisés, J. de Mécanique 14: 39-63 (1975)

[Kolmogorov & Fomin 1970] A.N. Kolmogorov, S.V. Fomin, *Introductory Real Analysis*, Dover Publications, New York 1970

[Lanczos 1949] C. Lanczos, *The Variational Principles of Mechanics*, University of Toronto Press 1949. Reprint: Dover Publications, New York 1986

[Lucchesi et al. 2006] M. Lucchesi, M. Šilhavý, N. Zani, A new class of equilibrated stress fields for no-tension bodies, J. Mech. Mater. Struct. 1: 503-539 (2006)

[Lucchesi et al. 2009] M. Lucchesi, M. Šilhavý, N. Zani, Equilibrated divergence measure stress tensor fields for heavy masonry bodies, Eur. J. Mech. Mater. A/28: 223-232 (2009)

[Marzocchi & Musesti 2003] A. Marzocchi, A. Musesti, The Cauchy stress theorem for bodies with finite perimeter, Rend. Sem. Mat. Univ. Padova 109: 1-11 (2003)

[Mindlin 1964] R.D. Mindlin, Micro-structure in linear elasticity, Arch. Ration. Mech. Anal. 16: 51-78 (1964)

[Mindlin & Eshel 1968] R.D. Mindlin, N.N. Eshel, On first strain-gradient theories in linear elasticity, Int. J. Solids Structures 4: 109-124 (1968)

[Noll 1959] W. Noll, The foundations of classical mechanics in the light of recent advances in continuum mechanics, in: *The Axiomatic Method, with Special Reference to Geometry and Physics*. North-Holland, Amsterdam (1959), pp. 266-281. Reprinted in: *The Foundations of Continuum Mechanics and Thermodynamics*, Selected Papers of W. Noll, Springer, Berlin 1974.

[Noll 1963] W. Noll, La mécanique classique, basée sur un axiome d'objectivité, in: *La Méthode Axiomatique dans les Mécaniques Classiques et Nouvelles*, Gauthier-Villars, Paris (1963), pp. 47-56. Reprinted in: *The Foundations of Continuum Mechanics and Thermodynamics*, Selected Papers of W. Noll, Springer, Berlin, 1974.

[Noll 1973] W. Noll, Lectures on the foundations of continuum mechanics and thermodynamics, Arch. Ration. Mech. Anal. 52: 62-92 (1973)

[Noll & Virga 1988] W. Noll, E.G. Virga, Fit regions and functions of bounded variation, Arch. Ration. Mech. Anal. 102: 1-21 (1988)

[Noll & Virga 1990] W. Noll, E.G. Virga, On edge interactions and surface tension, Arch. Ration. Mech. Anal. 111: 1-31 (1990)

[Podio Guidugli 2004] P. Podio Guidugli, Examples of concentrated contact interactions in simple bodies, J. of Elasticity 75: 167-186 (2004)

[Podio Guidugli & Vianello 2010] P. Podio Guidugli, M. Vianello, Hypertractions and hyperstresses convey the same mechanical information, Cont. Mech. Thermodyn. 22: 163-176 (2010)

[Šilhavý 1985] M. Šilhavý, The existence of the flux vector and the divergence theorem for general Cauchy fluxes, Arch. Ration. Mech. Anal. 90: 195-212 (1985)

[Šilhavý 1990] M. Šilhavý, On Cauchy's stress theorem, Rend. Mat. Acc. Lincei, ser. IX, 1: 259-263 (1990)

[Šilhavý 1991] M. Šilhavý, Cauchy's stress theorem and tensor fields with divergences in L^p, Arch. Ration. Mech. Anal. 116: 223-255 (1991)

[Šilhavý 2008] M. Šilhavý, Cauchy's stress theorem for stresses represented by measures, Cont. Mech. Thermodyn. 20: 75-96 (2008)

[Timoshenko 1953] S.P. Timoshenko, *History of Strength of Materials*, McGraw-Hill, New York 1953. Reprinted in: Dover Publications, New York 1983

[Toupin 1962] R.A. Toupin, Elastic materials with couple-stresses, Arch. Ration. Mech. Analysis 11: 385-414 (1962)

[Toupin 1964] R.A. Toupin, Theory of elasticity with couple-stress, Arch. Ration. Mech. Analysis 17: 85-112 (1964)

[Truesdell 1991] C.A. Truesdell, *A First Course in Rational Continuum Mechanics*, Vol. 1, 2nd Ed., Academic Press, Boston 1991

[Vol'pert & Hudjaev 1985] A.I. Vol'pert, S.I. Hudjaev, *Analysis in Classes of Discontinuous Functions and Equations of Mathematical Physics*, Nijhoff, Dordrecht 1985

[Ziemer 1983] W.P. Ziemer, Cauchy flux and sets of finite perimeter, Arch. Ration. Mech. Anal. 84: 189-201 (1983)

[Ziemer 1989] W.P. Ziemer, *Weakly Differentiable Functions*, Springer-Verlag, New York 1989