A RATIONAL APPROACH TO MICROPOLAR CONTINUA, WITH APPLICATION TO COSSERAT CONTINUA AND TO THEORIES OF BENDING OF PLATES AND BEAMS

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Abstract. Following a recently proposed approach to continua with microstructure, the theories of unconstrained and constrained Cosserat continua are reformulated. The proposed formulation only requires the specification of the form of the external power, plus some ad hoc indifference properties of the internal power.

From the model constructed in this way, by adding kinematical constraints which determine dimensional reduction, the classical equations for the bending of plates and beams are re-obtained in a surprisingly simple way.

1. Introduction

An alternative approach to continua with microstructure has been proposed in the papers [4, 5]. While the traditional formulations are based either on the balance laws of Euler and Cauchy, or, more recently, on the principle of virtual power [8, 9], the proposed approach is founded on a regularity property of the system of contact actions. Indeed, the assumption that the contact actions are *bounded Cauchy fluxes* [20] leads to an *equation of virtual power*, which states the equality between an external and an internal power. The external power is the product of the assumed kinematic variables by dual terms representing distance actions and contact actions, and the internal power is the product of *generalized internal forces* by *generalized deformations*. The two powers are not independent, as it is usually assumed. They are equivalent expressions of the same power.

The internal power is restricted by indifference requirements, whose form is dictated by the physical nature of the continuum. The resulting *reduced form* of the internal power specifies the generalized internal forces and the generalized deformations. That is, it determines the *structural properties* of the class of continua defined by the choice of the kinematic variables and by the specification of the indifference requirements. Within each class, relations between generalized internal forces and generalized deformations appropriate to specific materials are described by *constitutive equations*.

In this communication, attention is focused on the formulation of the equilibrium problem for continua with a particular type of microstructure, without considering any explicit constitutive equation. Due to the infinitesimal character of the virtual variations of the kinematic variables, the analysis is restricted to the incremental equilibrium problem from an arbitrary deformed configuration. For convenience, the current configuration is systematically taken as the reference configuration.

After introducing the equation of virtual power for micropolar continua, Section 2, in Section 3 the treatment is restricted to the Cosserat continua, which are characterized by a particular form of rotational indifference. Section 4 deals with the constrained theory, obtained by imposing the coincidence between the local rotations associated with the macroscopic and with the microscopic deformation.

The rest of the paper is devoted to the deduction of the classical bending theories of plates, Section 5, and beams, Section 6. This is done by the dimensional reduction obtained by imposing supplementary kinematic constraints to the three-dimensional Cosserat continuum. Thanks to the assumption of a bounded Cauchy flux, the deduction of plates and beams theories presented here is more simple and direct than those available in the literature [6, 7, 10, 15].

It must be observed that the plate and beam theories do not enjoy the same level of generality of the three-dimensional theory. Indeed, their range of application is limited by the assumed cylindrical shape of the body. In general, this shape is lost in a finite deformation. Therefore, it is impossible to preserve the advantages of cylindrical geometry in the incremental problem from a deformed configuration. Thus, the proposed formulation of the equilibrium problems for plates and beams holds only for small deformations from a deformed configuration which keeps the cylindrical geometry. A more general theory would require, at least, reference configurations with the shape of a shell or of a curved beam, respectively, with point-depending curvatures. This is out of the purposes of the present work.

Throughout the paper, all technical questions involving measure-theoretic concepts are omitted. For example, no regularity assumption for the shape of a body and of its parts is mentioned. In fact, a body is required to be a set of finite perimeter, and its boundary and the corresponding normal vector must be understood in the measure-thoretic sense. For a more complete presentation, the interested reader is addressed to the paper [5].

2. Micropolar continua

Following the definition given in [5], by a *micropolar continuum* we mean a continuum whose deformation is characterized by a vector field u, the *macroscopic displacement*, plus a finite number of vector fields d^α, the *directors*. The latter represent material directions which affect the body's response at the microscopic level. For example, the orientations of the crystalline lattice or the directions of crystal defects. The integral[1]

$$P_{ext}(\Pi, v, \nu^\alpha) = \int_\Pi (b(x)\cdot v(x) + \beta^\alpha(x)\cdot\nu^\alpha(x))\,dV + \int_{\partial\Pi}(s(x)\cdot v(x) + \sigma^\alpha(x)\cdot\nu^\alpha(x))\,dA \tag{2.1}$$

is the *virtual power* exerted on the portion Π of the body by virtual variations v and ν^α of u and d^α. The virtual displacements v act on given systems b of *body forces* and s of *surface tractions*, and the virtual velocities ν^α act on given systems β_α of *body microforces* and σ_α of *surface microtractions*. Under appropriate regularity assumptions,[2] it can be proved that the systems s and σ_α admit a volume density.

[1] Here and in the following, summation over repeated indices is assumed.

[2] That is, if s and all σ^α are the surface densities of bounded Cauchy fluxes, see [5], Sect. 4.

That is, there exist vector fields f, ϕ^α defined over the volume, such that

$$(2.2) \qquad \int_\Pi f(x)\, dV + \int_{\partial\Pi} s(x)\, dA \,=\, 0\,, \qquad \int_\Pi \phi^\alpha(x)\, dV + \int_{\partial\Pi} \sigma^\alpha(x)\, dA \,=\, 0\,,$$

for all parts Π of the body. These are the macroscopic and microscopic *pseudobalance equations*, respectively. By Noll's theorem on the dependence of the contact forces on the normal[3] and by Cauchy's tetrahedron theorem, from these equations the existence of second-order tensor fields T and Σ^α follows, such that

$$(2.3) \qquad s(x) \,=\, T(x)\, n\,, \qquad \sigma^\alpha(x) \,=\, \Sigma^\alpha(x)\, n\,,$$

where n is the exterior unit normal to Π. Substituting into the expression of the external power and using the divergence theorem, the right-hand side of (2.1) takes the form[4]

$$(2.4) \qquad \int_\Pi \big((\mathrm{div} T + b) \cdot v + T \cdot \nabla v + (\mathrm{div}\Sigma^\alpha + \beta^\alpha) \cdot \nu^\alpha + \Sigma^\alpha \cdot \nabla \nu^\alpha \big)\, dV\,.$$

This integral is called the *internal power*, and is denoted by $P_{int}(\Pi, v, \nu^\alpha)$. As pointed out in the Introduction, this is not a power of independently assumed internal forces. As a consequence of the pseudobalance equations, this is just *an alternative expression* of the external power. That is, external power and internal power *are two equivalent expressions of the same virtual power*, and the *equation of virtual power*

$$(2.5) \qquad P_{ext}(\Pi, v, \nu^\alpha) = P_{int}(\Pi, v, \nu^\alpha)$$

is in fact an identity, which holds for all bounded Cauchy fluxes.

The virtual power is subject to indifference requirements: it must be insensitive to rigid virtual velocities v, ν^α. What *rigid* means, depends on the physical nature of the order parameters. In the next Sections, definitions appropriate to unconstrained and to constrained Cosserat continua will be given.

The indifference restrictions produce the fundamental balance laws of mechanics, which are classically considered as postulates.[5] Specifically, the indifference to rigid translations produces the *balance law of linear momentum*, and the indifference to rigid rotations produces the *balance law of angular momentum.* When substituted in the expression (2.4) of the virtual power, the two laws determine which ones of the terms $(\mathrm{div}T + b)$, T, $(\mathrm{div}\Sigma^\alpha + \beta^\alpha)$, Σ^α are *generalized forces* and which ones are *reactions.* The difference is that, as shown in the following Sections, the reactions are determined by the generalized forces, while the latter are related to the *generalized deformations* by constitutive equations.

The differential system made of the balance equations plus the constitutive equations, completed by appropriate boundary conditions, forms the *incremental equilibrium problem for the micropolar continuum.*[6] The rest of the paper is devoted to the formulation of this problem for three-, two-, and one-dimensional Cosserat

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[3] Noll [16]. For a proof in the context of measure theory see [5], Theorem 4.3.

[4] From here onwards, the argument x is omitted for simplicity.

[5] Noll [17]. The deduction of the balance laws from indifference conditions is a special case of Noether's theorem on the correspondence between indifference properties of a functional and conservation laws, see e.g. [12], p. 403.

[6] The adjective *incremental* refers to the fact that the equilibrium problem stated below determines the response to a small perturbation of the data, starting from a known deformed equilibrium configuration.

continua.

3. The Cosserat continuum

A Cosserat continuum is a micropolar continuum characterized by an orthonormal triplet of directors d^α, which remains orthonormal in all deformed configurations. In a virtual deformation, at any point x the triplet undergoes a rigid rotation described by a vector $\omega(x)$. The virtual velocity is

$$\nu^\alpha(x) = \omega(x) \times d^\alpha(x)\,, \tag{3.1}$$

and, omitting again the argument x, the corresponding virtual powers are

$$\begin{aligned} \beta^\alpha \cdot \nu^\alpha &= \beta^\alpha \cdot \omega \times d^\alpha = d^\alpha \times \beta^\alpha \cdot \omega\,, \\ \sigma^\alpha \cdot \nu^\alpha &= \sigma^\alpha \cdot \omega \times d^\alpha = d^\alpha \times \sigma^\alpha \cdot \omega\,. \end{aligned} \tag{3.2}$$

By defining the *body couple* and the *surface couple*

$$c = d^\alpha \times \beta^\alpha\,, \qquad m = d^\alpha \times \sigma^\alpha\,, \tag{3.3}$$

with summation over the repeated indices α, the external power takes the form

$$P_{ext}(\Pi, v, \omega) = \int_\Pi (b \cdot v + c \cdot \omega)\, dV + \int_{\partial^*\Pi} (s \cdot v + m \cdot \omega)\, dA\,. \tag{3.4}$$

Comparison with (2.1) shows that a Cosserat continuum is a micropolar continuum with a single order parameter ω. By consequence, there is only one microscopic pseudobalance equation $(2.2)_2$, and equations (2.3) are replaced by

$$s = Tn\,, \qquad m = Mn\,, \tag{3.5}$$

with T the *Cauchy stress tensor* and M the *couple-stress tensor*. The internal power (2.4) then takes the form

$$P_{int}(\Pi, v, \omega) = \int_\Pi \Big((\mathrm{div}T + b) \cdot v + T \cdot \nabla v + (\mathrm{div}M + c) \cdot \omega + M \cdot \nabla\omega \Big)\, dV. \tag{3.6}$$

For a Cosserat continuum, the *rigid* virtual velocities are the rigid translations of the body, $v(x) = a$, and the simultaneous rigid rotations of the body and of the directors

$$v(x) = a \times x\,, \qquad \omega(x) = a\,. \tag{3.7}$$

Therefore, the indifference conditions are

$$P_{int}(\Pi, a, 0) = 0\,, \qquad P_{int}(\Pi, a \times x, a) = 0\,. \tag{3.8}$$

They must be satisfied for all vectors a. Using the arbitrariness of Π, the translational indifference condition provides the *balance equation of linear momentum*[7]

$$\mathrm{div}T + b = 0\,. \tag{3.9}$$

Moreover, denoting by t the vector associated with the skew-symmetric part of T

$$t_i = \tfrac{1}{2}\, e_{kij} T_{kj}\,, \tag{3.10}$$

the identity

$$T \cdot \nabla(a \times x) = 2\, t \cdot a\,, \qquad T_{kj}\, e_{kih}\, a_i\, x_{h,j} = T_{kj}\, e_{kij}\, a_i = 2\, t_i\, a_i\,,$$

[7] Inertia forces can be included as particular body forces and microforces, see [17], Sect. 7.

holds. Then the rotational indifference condition requires that

$$\operatorname{div}M + c + 2\,t = 0\,, \qquad M_{ij,j} + c_i + e_{kij}T_{kj} = 0\,. \tag{3.11}$$

This is the form taken by the *balance equation of angular momentum* for the Cosserat continuum. It says that the Cauchy stress is not symmetric, and that its skew-symmetric part is determined by M and c.

Equations (3.9) and (3.11) are the equilibrium equations at the internal points of the body Ω, and equations (3.5) are the *boundary conditions of traction*, to be imposed at the free part of the boundary, that is, at the boundary points of Ω at which the contact forces s and the contact couples m are prescribed. At the remaining boundary points of Ω, the *boundary conditions of place* must be prescribed. They consist in prescribing the values of v and ω

$$v(x) = \hat{v}(x)\,, \qquad \omega(x) = \hat{\omega}(x)\,. \tag{3.12}$$

To complete the formulation of the equilibrium problem, it is necessary to prescribe constitutive equations between generalized forces and generalized deformations. To define these objects, let us introduce the decompositions of T and ∇v into the sums of their symmetric and skew-symmetric parts

$$T = T^S + T^W\,, \quad \nabla v = \nabla^S v + \nabla^W v\,, \quad T\cdot\nabla v \,=\, T^S\cdot\nabla^S v + T^W\cdot\nabla^W v\,. \tag{3.13}$$

Using the balance equations and the identity

$$T^W\cdot\nabla^W v = t\cdot\operatorname{curl} v\,. \tag{3.14}$$

which is a consequence of (3.10), the internal power (3.6) can be given the form

$$P_{int}(\Pi, v, \omega) = \int_\Pi \big(T^S\cdot\nabla^S v + t\cdot(\operatorname{curl} v - 2\,\omega) + M\cdot\nabla\omega\big)\,dV. \tag{3.15}$$

It shows that the generalized forces are T^S, t and M, and that the corresponding generalized deformations are $\nabla^S v$, $2\,\varphi$ and $\nabla\omega$, where

$$\varphi \,=\, \tfrac{1}{2}\operatorname{curl}\, v - \omega \tag{3.16}$$

is the virtual relative rotation between the body and the triad of the directors. Thus, for an elastic Cosserat continuum the constitutive equations have the form

$$\begin{gathered} T^S = \hat{T}^S(\nabla^S v, 2\,\varphi, \nabla\omega)\,, \quad t = \hat{t}(\nabla^S v, 2\,\varphi, \nabla\omega)\,, \\ M = \hat{M}(\nabla^S v, 2\,\varphi, \nabla\omega)\,. \end{gathered} \tag{3.17}$$

Using the identity

$$\operatorname{curl} t = -\operatorname{div}T^W\,, \tag{3.18}$$

which follows from (3.10), the equilibrium equation (3.9) can be expressed in terms of the generalized forces

$$\operatorname{div}T^S - \operatorname{curl} t + b = 0\,, \qquad T^S_{ij,j} - e_{ijk}\,t_{k,j} + b_i = 0\,. \tag{3.19}$$

For the boundary conditions of traction $(3.5)_1$, again from (3.10),

$$s_i = T_{ij}\,n_j = (T^S_{ij} - e_{ijk}t_k)\,n_j\,. \tag{3.20}$$

In a local orthonormal reference system $\{e^i\}$ with $e^3 = n$ we have

$$s_i = T_{in} = T^S_{in} + e_{ikn}t_k\,, \tag{3.21}$$

and, writing separately the normal and the tangent components and setting

$$e_{\alpha\beta n} \,=\, e_{\alpha\beta}\,, \tag{3.22}$$

we finally get

$$T^S_{nn} = s_n\,, \qquad T^S_{\alpha n} + e_{\alpha\beta} t_\beta = s_\alpha\,. \tag{3.23}$$

Similarly, from conditions $(3.5)_2$ we get

$$M_{nn} = m_n\,, \qquad M_{\alpha n} = m_\alpha\,. \tag{3.24}$$

In their final form, the equations of the incremental equilibrium problem for the elastic Cosserat continuum are collected in Table 1 at the end of the paper.

4. The constrained Cosserat continuum

In view of the deduction of engineering theories for plates and beams, consider the Cosserat continuum subject to the kinematic constraint[8]

$$\omega \;=\; \tfrac{1}{2}\,\mathrm{curl}\, v\,. \tag{4.1}$$

This constraint imposes that the relative rotation (3.16) be zero at all points of the body and in all virtual deformations. By consequence, the external power (3.4) becomes[9]

$$P_{ext}(\Pi, v) = \int_\Pi (b\cdot v + \tfrac{1}{2}\, c\cdot \mathrm{curl}\, v)\, dV + \int_{\partial\Pi} (s\cdot v + \tfrac{1}{2}\, m\cdot \mathrm{curl}\, v)\, dA\,. \tag{4.2}$$

For sufficiently regular contact actions s and m the pseudobalance equations (2.2) hold. Then there are second-order tensor fields T, M for which equations (3.5) are satisfied. The expression (3.6) of the internal power follows, with ω replaced by $\frac{1}{2}\mathrm{curl}\, v$, and the indifference conditions (3.8) provide the balance equations (3.9) and (3.11), and the reduced internal power

$$P_{int}(\Pi, v) = \int_\Pi (T^S\cdot \nabla^S v + M\cdot \tfrac{1}{2}\,\nabla \mathrm{curl}\, v)\, dV. \tag{4.3}$$

The generalized forces are now T^S and M, and the generalized deformations are $\nabla^S v$ and $\frac{1}{2}\,\mathrm{curl}\, v$. By consequence, the constitutive equations are

$$T^S = \hat{T}^S(\nabla^S v, \tfrac{1}{2}\,\mathrm{curl}\, v)\,, \qquad M = \hat{M}(\nabla^S v, \tfrac{1}{2}\,\mathrm{curl}\, v)\,. \tag{4.4}$$

The kinematic variables v and $\mathrm{curl}\, v$ are not independent, and in both expressions of the power the volume terms involving $\mathrm{curl}\, v$ can be eliminated using the divergence

[8] Toupin [23], Sect. 11.

[9] Due to the constraint (4.1), the only variable left in the expression of the virtual power is v. However, the presence of a hidden variable ω is revealed by the expression of the external power, which has an extra term with respect to the power of a classical continuum. In the terminology introduced in [2], this is a *continuum with latent microstructure*.

theorem

$$\begin{aligned}P_{ext}(\Pi, v) &= \int_\Pi (b_i\, v_i + \tfrac{1}{2}\, c_k\, e_{ikj} v_{i,j})\, dV + \int_{\partial\Pi} (s_i v_i + \tfrac{1}{2}\, m_k\, e_{ikj} v_{i,j})\, dA \\ &= \int_\Pi (b_i - \tfrac{1}{2}\, c_{k,j}\, e_{ikj})\, v_i\, dV + \int_{\partial\Pi} \big((s_i + \tfrac{1}{2}\, c_k\, e_{ikj}\, n_j)\, v_i + \tfrac{1}{2}\, m_k\, e_{ikj} v_{i,j}\big)\, dA\,,\end{aligned}$$

$$\begin{aligned}P_{int}(\Pi, v) &= \int_\Pi \big(T^S_{ij}\, v_{i,j} + \tfrac{1}{2}\, M_{kh} e_{ikj} v_{i,jh}\big)\, dV \\ &= \int_\Pi \big(-T^S_{ij,j} + \tfrac{1}{2}\, M_{kh,hj}\, e_{ikj}\big)\, v_i\, dV \\ &\qquad + \int_{\partial\Pi} \big((T^S_{ij} - \tfrac{1}{2}\, M_{kh,h} e_{ikj})\, v_i n_j + \tfrac{1}{2}\, M_{kh} e_{ikj} v_{i,j} n_h\big)\, dA\,.\end{aligned} \tag{4.5}$$

Equating the two expressions of the power obtained in this way and using the arbitrariness of v_k, from the volume integrals we get

$$T^S_{ij,j} + b_i + \tfrac{1}{2}\, e_{ijk}\, (M_{kh,hj} + c_{k,j}) = 0\,. \tag{4.6}$$

This is a combination of the equilibrium equations (3.9), (3.11) of the unconstrained continuum. Indeed, by (3.11) and (3.19),

$$\operatorname{curl}(\operatorname{div} M + c) = -2 \operatorname{curl} t = 2 \operatorname{div} T^W = -2\,(\operatorname{div} T^S + b)\,, \tag{4.7}$$

and this is exactly equation (4.6). Moreover, equating the surface integrals on the right sides of (4.5) we get

$$\int_{\partial\Pi} \big((s_i - T^S_{ij} n_j + \tfrac{1}{2}(c_k + M_{kh,h})\, e_{ikj}\, n_j)\, v_i + \tfrac{1}{2}(m_k - M_{kh} n_h)\, e_{ikj} v_{i,j}\big)\, dA = 0\,. \tag{4.8}$$

Of the gradient $v_{i,j}$, only the normal component is an independent variable, the tangential components being determined by the boundary values of v_i. Therefore, on $\partial\Pi$ we take a local orthonormal reference system $\{e^\alpha, e^\beta, n\}$, with n the exterior unit normal. After rewriting the last boundary term with separated normal and tangential components[10]

$$\begin{aligned}(m_k - M_{kh} n_h)\, e_{ikj} v_{i,j} &= (m_k - M_{kn})\, (e_{ikn} v_{i,n} + e_{ik\alpha} v_{i,\alpha}) \\ &= (m_\beta - M_{\beta n})\, e_{\alpha\beta} v_{\alpha,n} + (m_k - M_{kn})\, e_{ik\beta} v_{i,\beta}\,,\end{aligned}$$

a further application of the divergence theorem

$$\int_{\partial\Pi} (m_k - M_{kn})\, e_{ik\beta} v_{i,\beta}\, dA = -\int_{\partial\Pi} (m_k - M_{kn})_{,\beta}\, e_{ik\beta} v_i\, dA\,, \tag{4.9}$$

and substitution into (4.8) yields

$$\begin{aligned}\int_{\partial\Pi} \big((s_i - T^S_{in} + \tfrac{1}{2}\,(c_k + M_{kh,h})\, e_{ikn} - \tfrac{1}{2}\,(m_k - M_{kn})_{,\beta}\, e_{ik\beta})\, v_i \\ + \tfrac{1}{2}\,(m_\beta - M_{\beta n})\, e_{\alpha\beta}\, v_{\alpha,n}\big)\, dA = 0\,.\end{aligned} \tag{4.10}$$

This equation is satisfied by imposing the boundary conditions of place

$$v_\alpha(x) = \hat{v}_\alpha(x)\,, \quad v_n(x) = \hat{v}_n(x)\,, \quad v_{\alpha,n}(x) = \hat{v}_{\alpha,n}(x)\,, \tag{4.11}$$

at the constrained part of the boundary, and the boundary conditions of traction

$$\begin{gathered}T^S_{\alpha n} + \tfrac{1}{2}\, e_{\alpha\beta}(M_{nn,\beta} - M_{\beta h,h}) = s_\alpha + \tfrac{1}{2}\, e_{\alpha\beta}(c_\beta + m_{n,\beta})\,, \\ T^S_{nn} + \tfrac{1}{2}\, e_{\alpha\beta} M_{\beta n,\alpha} = s_n + \tfrac{1}{2}\, e_{\alpha\beta} m_{\beta,\alpha}\,, \\ M_{\alpha n} = m_\alpha\,.\end{gathered} \tag{4.12}$$

[10] In the second equality, $e_{\alpha\beta} = e_{\alpha\beta n}$.

at the free part of the boundary. There is no condition on the normal derivative $v_{n,n}$ and on the corresponding surface traction. That is, the boundary conditions to be satisfied are five, instead of the six of the unconstrained theory.[11]

Moreover, the last condition $M_{\alpha n} = m_\alpha$ implies $M_{\beta n,\alpha} = m_{\beta,\alpha}$. Therefore, the second boundary condition simplifies to[12]

$$T^S_{nn} = s_n \,. \tag{4.13}$$

The equations of the equilibrium problem for the constrained Cosserat continuum are collected in Table 1 at the end of the paper.

5. Plate theories

A plate can be viewed as a body of a cylindrical shape, made of a Cosserat continuum subjected to the kinematic constraints

$$v(x) = v_3(x_1, x_2)\, e^3, \qquad \omega(x) = \omega_\alpha(x_1, x_2)\, e^\alpha, \tag{5.1}$$

with e^3 the direction of the cylinder's axis. The constraints require that, at all points x, the displacement $v(x)$ be parallel to e^3 and the rotation $\omega(x)$ of the directors be about an axis orthogonal to e^3. Therefore, the three-dimensional vectors v and ω degenerate into a scalar and into a two-dimensional vector, respectively. The same do the associated vectors b, s and c, m. Then, the external power takes the form[13]

$$P_{ext}(\Pi, v, \omega) = \int_\Pi (b_3 v_3 + c_\alpha \omega_\alpha)\, dV + \int_{\partial\Pi} (s_3 v_3 + m_\alpha \omega_\alpha)\, dA \,. \tag{5.2}$$

The constraints (5.1) also require that both v and ω be independent of x_3. Therefore, the body can be identified with the cylinder's cross section. With this dimensional reduction, the parts Π of the body reduce to plane surfaces, the volume elements dV reduce to area elements, and the area elements dA reduce to line elements. In spite of this, we prefer to keep the notation dV and dA.

The pseudobalance equations formally coincide with equations (2.2). However, due to dimensional reduction, the stress tensor T_{ij} degenerates into the vector Q_α of the *internal shearing forces*, and the couple-stress tensor M_{ij} degenerates into the 2×2 tensor of the *internal moments* $M_{\alpha\beta}$. Therefore, equations (2.3) take the form

$$s_3 = Q_\alpha n_\alpha \,, \qquad m_\alpha = M_{\alpha\beta} n_\beta \,. \tag{5.3}$$

The component b_3 of the body force is now viewed as a transverse load q. Thus, the internal power (2.4) takes the form

$$P_{int}(\Pi, v, \omega) = \int_\Pi \big((Q_{\alpha,\alpha} + q)\, v_3 + Q_\alpha v_{3,\alpha} + (M_{\alpha\beta,\beta} + c_\alpha)\, \omega_\alpha + M_{\alpha\beta} \omega_{\alpha,\beta} \big)\, dV. \tag{5.4}$$

In the indifference requirements (3.8), a is now any vector orthogonal to e^3. Accordingly, the balance equations (3.9), (3.11) become

$$Q_{\alpha,\alpha} + q = 0 \,, \qquad M_{\alpha\beta,\beta} + c_\alpha + e_{\alpha\beta}\, Q_\beta = 0 \,, \tag{5.5}$$

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[11] See Schaefer [19].

[12] In the more general context of second-gradient continua, a similar simplification was made by Bleustein [1].

[13] From here onwards, Greek indices run from 1 to 2, and Latin indices run from 1 to 3.

and substitution into (5.4) yields

$$P_{int}(\Pi, v, \omega) = \int_{\Pi} \big(Q_\alpha (v_{3,\alpha} + e_{\alpha\beta}\omega_\beta) + M_{\alpha\beta}\omega_{\alpha,\beta} \big)\, dV. \tag{5.6}$$

Thus, the generalized forces are Q_α and $M_{\alpha\beta}$, and the associated generalized deformation are the rotation gradient $\nabla\omega$ and

$$\varphi_\alpha = v_{3,\alpha} + e_{\alpha\beta}\omega_\beta\,, \tag{5.7}$$

which is the two-dimensional counterpart of the relative rotation (3.16). The constitutive equations for the elastic plate are

$$Q_\alpha = \hat{Q}_\alpha(\varphi, \nabla\omega)\,, \qquad M_{\alpha\beta} = \hat{M}_{\alpha\beta}(\varphi, \nabla\omega)\,. \tag{5.8}$$

Equations (5.5) and (5.8), plus the boundary conditions of place

$$v_3(x) = \hat{v}_3(x)\,, \qquad \omega_\alpha(x) = \hat{\omega}_\alpha(x)\,, \tag{5.9}$$

on the constrained part of $\partial\Omega$ and conditions (5.3) on the free part, rewritten as

$$s_3 = Q_n\,, \qquad m_\alpha = M_{\alpha n}\,, \tag{5.10}$$

form the equilibrium problem for the *Reissner theory of plates* [18]. The equations have been deduced from those of the three-dimensional Cosserat continuum, using the dimensional reduction produced by the kinematic constraints (5.1).

Just as the constrained Cosserat continuum was obtained by introducing the kinematic constraint (4.1), the *Kirchhoff-Love theory of plates* [11, 13] can be deduced from Reissner's theory by imposing the kinematic constraint

$$\omega_\alpha \;=\; e_{\alpha\beta} v_{3,\beta}\,, \tag{5.11}$$

which requires that the relative rotation (5.7) be zero. With this restriction, the external power (5.2) reduces to

$$P_{ext}(\Pi, v) = \int_{\Pi} (b_3 v_3 + c_\alpha e_{\alpha\beta} v_{3,\beta})\, dV + \int_{\partial\Pi} (s_3 v_3 + m_\alpha e_{\alpha\beta} v_{3,\beta})\, dA\,. \tag{5.12}$$

Using equations (5.3) and setting $b_3 = q$ and[14]

$$c^*_\beta = c_\alpha e_{\alpha\beta}\,, \quad m^*_\beta = m_\alpha e_{\alpha\beta}\,, \quad M^*_{\beta\gamma} = M_{\alpha\gamma} e_{\alpha\beta}\,, \tag{5.13}$$

after an integration by parts, the internal power

$$P_{int}(\Pi, v) = \int_{\Pi} \big((Q_{\beta,\beta} + q)\, v_3 + Q_\beta v_{3,\beta} + (M^*_{\beta\gamma,\gamma} + c^*_\beta)\, v_{3,\beta} + M^*_{\beta\gamma} v_{3,\beta\gamma} \big)\, dV \tag{5.14}$$

is obtained. The indifference requirements (3.8) now provide the balance equations

$$Q_{\beta,\beta} + q = 0\,, \qquad M^*_{\beta\gamma,\gamma} + c^*_\beta + Q_\beta = 0\,, \tag{5.15}$$

and the internal power reduces to

$$P_{int}(\Pi, v) = \int_{\Pi} M^*_{\alpha\beta} v_{3,\alpha\beta}\, dV. \tag{5.16}$$

Thus, there is a single generalized force, $M^*_{\alpha\beta}$, and the associated generalized deformation is $v_{3,\alpha\beta}$. By the symmetry of the second derivative, only the symmetric

[14] The moments M^*_{11} and M^*_{22} are bending moments, and M^*_{12} and M^*_{21} are twisting moments. They are the moments currently used in plate theories, see, e.g., [22], Sect. 21, Fig. 47.

part M^{*S} of M^* contributes to the power. That is, M^{*S} is the effective generalized force. Accordingly, the constitutive equation has the form

$$M^{*S}_{\alpha\beta} = \hat{M}^{*S}_{\alpha\beta}(\nabla\nabla v_3)\,. \tag{5.17}$$

To get appropriate boundary conditions, we follow the same procedure adopted for the three-dimensional constrained continuum. First, using the divergence theorem, we eliminate from the volume integrals in (5.12) and (5.16) the terms involving the derivatives of v_3

$$\begin{aligned} P_{ext}(\Pi, v) &= \int_\Pi (q\, v_3 + c^*_\alpha\, v_{3,\alpha})\, dV + \int_{\partial\Pi} (s_3 v_3 + m^*_\alpha\, v_{3,\alpha})\, dA \\ &= \int_\Pi (q - c^*_{\alpha,\alpha})\, v_3\, dV + \int_{\partial\Pi} \big((s_3 + c^*_\alpha\, n_\alpha)\, v_3 + m^*_\alpha v_{3,\alpha}\big)\, dA\,, \\ P_{int}(\Pi, v) &= \int_\Pi M^*_{\alpha\beta,\alpha\beta} v_3\, dV + \int_{\partial\Pi} \big(M^*_{\alpha\beta} v_{3,\alpha} n_\beta - M^*_{\alpha\beta,\beta} v_3 n_\alpha\big)\, dA\,. \end{aligned} \tag{5.18}$$

Then, comparing the volume terms and observing that $M^*_{\alpha\beta,\alpha\beta} = M^{*S}_{\alpha\beta,\alpha\beta}$, we get the field equation

$$M^{*S}_{\alpha\beta,\alpha\beta} + c^*_{\alpha,\alpha} - q = 0\,, \tag{5.19}$$

and comparing the boundary terms we have

$$\int_{\partial\Pi} \big((M^*_{\alpha\beta,\beta} n_\alpha + s_3 + c^*_\alpha\, n_\alpha)\, v_3 - (M^*_{\alpha\beta} n_\beta - m^*_\alpha)\, v_{3,\alpha}\big)\, dA = 0\,. \tag{5.20}$$

Like in the three-dimensional constrained theory, only the component of $v_{3,\alpha}$ normal to the boundary is independent, the tangential component being determined by the values of v_3 at the boundary. Therefore, keeping e^3 in the direction of the cylinder's axis, we take a local reference system with e^1, e^2 coincident with the outward normal n and the tangent vector τ to the lateral surface, respectively. Then, with a further use of the divergence theorem, the last equation transforms as follows

$$\begin{aligned} 0 &= \int_{\partial\Pi} \big((M^*_{n\beta,\beta} + s_3 + c^*_n)\, v_3 - (M^*_{\tau n} - m^*_\tau)\, v_{3,\tau} - (M^*_{nn} - m^*_n)\, v_{3,n}\big)\, dA \\ &= \int_{\partial\Pi} \big((M^*_{n\beta,\beta} + M^*_{\tau n,\tau} + s_3 + c^*_n - m^*_{\tau,\tau})\, v_3 - (M^*_{nn} - m^*_n)\, v_{3,n}\big)\, dA\,. \end{aligned} \tag{5.21}$$

At each point on the constrained part of the boundary are prescribed the conditions of place

$$v_3(x) = \hat{v}_3(x)\,, \qquad v_{3,n}(x) = \hat{v}_{3,n}(x)\,, \tag{5.22}$$

and at each point on the free part of the boundary are prescribed the conditions of traction

$$M^*_{n\beta,\beta} + M^*_{\tau n,\tau} + s_3 + c^*_n - m^*_{\tau,\tau} = 0\,, \quad M^*_{nn} - m^*_n = 0\,.$$

In the first condition,

$$M^*_{n\beta,\beta} + M^*_{\tau n,\tau} \;=\; M^*_{nn,n} + M^*_{n\tau,\tau} + M^*_{\tau n,\tau} \;=\; M^{*S}_{nn,n} + 2\, M^{*S}_{n\tau,\tau}\,,$$

and because $M^*_{nn} = M^{*S}_{nn}$, the conditions of traction become

$$M^{*S}_{nn,n} + 2\, M^{*S}_{n\tau,\tau} + s_3 + c^*_n - m^*_{\tau,\tau} \,=\, 0\,, \qquad M^{*S}_{nn} - m^*_n \,=\, 0\,. \tag{5.23}$$

Like in the three-dimensional case, the number of the boundary conditions is smaller in the constrained model. Indeed, the three scalar conditions (5.10) reduce to the two conditions (5.23).

6. Beam theories

In classical beam theories, a beam is viewed as a body with a cylindrical shape, made of a Cosserat continuum subjected to the kinematic constraint

$$v(x) = v_\alpha(x_3)\, e^\alpha\,, \qquad \omega(x) = \omega_\alpha(x_3)\, e^\alpha\,, \tag{6.1}$$

with e^3 the direction of the cylinder's axis. Under such constraint, each cross section of the cylinder undergoes a rigid translation v_α orthogonal to e^3, and the triple of the directors undergoes a rigid rotation ω about an axis orthogonal to e^3. Thus, the body can be reduced to the cylinder's axis, the parts Π of the body reduce to intervals (a,b), and the boundary $\partial\Pi$ reduces to the endpoints a, b. The external power takes the form

$$P_{ext}((a,b),v,\omega) = \int_a^b (q_\alpha v_\alpha + c_\alpha \omega_\alpha)\, dx_3 + \big(P_\alpha v_\alpha + C_\alpha \omega_\alpha\big)_b + \big(P_\alpha v_\alpha + C_\alpha \omega_\alpha\big)_a\,, \tag{6.2}$$

where q_α and c_α are distributed forces and couples per unit length, and P_α and C_α are the concentrated couples and forces representing the contact actions between (a,b) and the rest of the beam.

The pseudobalance equations now imply the existence of two fields of internal forces, the *shearing force* Q_α and the *bending moment* M_α, such that

$$P_\alpha = Q_\alpha n\,, \qquad C_\alpha = M_\alpha n\,. \tag{6.3}$$

The exterior unit normal n to the cross section is e^3 at $x_3 = b$ and $-e^3$ at $x_3 = a$. Substituting into (6.2) and integrating by parts one gets the expression of the internal power

$$P_{int}((a,b),v,\omega) = \int_a^b \big((q_\alpha + Q'_\alpha)\, v_\alpha + Q_\alpha v'_\alpha + (c_\alpha + M'_\alpha)\, \omega_\alpha + M_\alpha \omega'_\alpha\big)\, dx_3\,, \tag{6.4}$$

where a prime denotes differentiation with respect to x_3. The indifference conditions have again the form (3.8), with a a vector parallel to e^3. They express indifference of the power to rigid translations of the segment (a,b) in the direction of e^3, and to simultaneous rigid rotations of (a,b) and of the triple of the directors about any axis orthogonal to e^3. They provide the balance equations

$$q_\alpha + Q'_\alpha = 0\,, \qquad c_\alpha + M'_\alpha - e_{\alpha\beta}\, Q_\beta = 0\,, \tag{6.5}$$

which, substituted into (6.4), yield the internal power

$$P_{int}((a,b),v,\omega) = \int_a^b \big(Q_\alpha(v'_\alpha - e_{\alpha\beta}\,\omega_\beta) + M_\alpha \omega'_\alpha\big)\, dx_3\,. \tag{6.6}$$

This equation shows that for the beam model the generalized internal forces are M_α and Q_α, and the corresponding generalized deformations are ω'_α and

$$\varphi_\alpha \;=\; v'_\alpha - e_{\alpha\beta}\,\omega_\beta\,. \tag{6.7}$$

The latter is the one-dimensional counterpart of the relative rotation (3.16). For an elastic material, the constitutive equations are of the form

$$Q_\alpha = \hat{Q}_\alpha(\varphi,\omega')\,, \qquad M_\alpha = \hat{M}_\alpha(\varphi,\omega')\,. \tag{6.8}$$

Together with the equilibrium equations (6.5), the boundary conditions of place

$$v_3(l) = v_{3l}\,, \quad v_3(0) = v_{30}\,, \quad \omega_\alpha(l) = \omega_{\alpha l}\,, \quad \omega_\alpha(0) = \omega_{\alpha 0}\,, \tag{6.9}$$

and the boundary conditions of traction, they form the equilibrium problem for the *Timoshenko beam theory*. By (6.3) with $n_3 = \pm 1$, the boundary conditions of traction are

$$Q_\alpha(l) = P_{\alpha l}\,, \quad Q_\alpha(0) = -P_{\alpha 0} \tag{6.10}$$

for the concentrated forces $P_{\alpha l}$, $P_{\alpha 0}$ applied at the endpoints of the beam, and

$$M_\alpha(l) = C_{\alpha l}\,, \quad M_\alpha(0) = -C_{\alpha 0}\,, \tag{6.11}$$

for the concentrated couples applied at the same points.

The *Euler-Bernoulli beam theory* is obtained by imposing the kinematic constraint

$$\omega_\alpha = -e_{\alpha\beta}\, v'_\beta\,, \tag{6.12}$$

which consists in assuming that the relative rotation (6.7) is identically zero. Under this constraint, the external power (6.2) becomes

$$\begin{aligned} P_{ext}((a,b),v) = \int_a^b (q_\alpha v_\alpha - c_\alpha\, e_{\alpha\beta} v'_\beta)\, dx_3 \\ + \big(P_\alpha v_\alpha - C_\alpha\, e_{\alpha\beta} v'_\beta\big)_b + \big(P_\alpha v_\alpha - C_\alpha\, e_{\alpha\beta} v'_\beta\big)_a\,, \end{aligned} \tag{6.13}$$

and using equations (6.3) and the equilibrium equations (6.5) the internal power (6.6) reduces to

$$P_{int}((a,b),v) = -\int_a^b M_\alpha\, e_{\alpha\beta} v''_\beta\, dx_3\,. \tag{6.14}$$

The unique generalized force is M_α, the generalized deformation is the *curvature vector* $\kappa_\alpha = -e_{\alpha\beta} v''_\beta$, and the constitutive equation is

$$M_\alpha = \hat{M}_\alpha(\kappa)\,. \tag{6.15}$$

Moreover, integrating by parts equations (6.13), (6.14), we get

(6.16)

$$\begin{aligned} P_{ext}((a,b),v) &= \int_a^b (q_\alpha - e_{\alpha\beta} c'_\beta)\, v_\alpha\, dx_3 \\ &\quad + \big((P_\alpha + e_{\alpha\beta} c_\beta)\, v_\alpha - C_\alpha\, e_{\alpha\beta}\, v'_\beta\big)_b + \big((P_\alpha - e_{\alpha\beta} c_\beta)\, v_\alpha - C_\alpha\, e_{\alpha\beta}\, v'_\beta\big)_a\,, \\ P_{int}((a,b),v) &= \int_a^b M_\beta\, e_{\alpha\beta} v''_\alpha\, dx_3 \\ &= \int_a^b M''_\beta\, e_{\alpha\beta} v_\alpha\, dx_3 + \big(M_\beta\, e_{\alpha\beta}\, v'_\alpha - M'_\beta e_{\alpha\beta}\, v_\alpha\big)_b - \big(M_\beta\, e_{\alpha\beta}\, v'_\alpha - M'_\beta e_{\alpha\beta}\, v_\alpha\big)_a. \end{aligned}$$

Comparing the two integrals, from the arbitrariness of v_α we obtain

$$M''_\alpha + c'_\alpha + e_{\alpha\beta} q_\beta = 0\,, \tag{6.17}$$

which is a combination of the equilibrium equations (6.5). Moreover, at the endpoint $x = l$,

$$M'_\alpha(l) + c_\alpha(l) = e_{\alpha\beta} P_{\beta l}\,, \qquad M_\alpha(l) = C_{\alpha l}\,, \tag{6.18}$$

and at the endpoint $x = 0$,

$$M'_\alpha(0) + c_\alpha(0) = -e_{\alpha\beta} P_{\beta 0}\,, \quad M_\alpha(0) = -C_{\alpha 0}\,. \tag{6.19}$$

The boundary conditions of place consist in prescribing the displacements v_α and the rotations v'_α

$$v_\alpha(l) = v_{\alpha l}\,, \quad v_\alpha(0) = v_{\alpha 0}\,, \quad v'_\alpha(l) = v'_{\alpha l}\,, \quad v'_\alpha(0) = v'_{\alpha 0}\,. \tag{6.20}$$

7. Conclusion

In the preceding Sections, the equilibrium problems for some classical theories of plates and beams have been deduced formally from the equilibrium problem of the three-dimensional Cosserat continuum, unconstrained and constrained, by imposing appropriate kinematic constraints to the body's deformation. The field equations and the boundary conditions of the considered problems are collected in Table 1. For an easier comparison, all equations are written in components.

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TABLE 1. The incremental equilibrium problems for the 3-, 2-, 1-dimensional, ordinary and constrained Cosserat continua. In each box, the equilibrium equations, the constitutive equations, and the boundary conditions of place and of traction are given in sequence. The left column gives the place of each equation in the paper

3D: Unconstrained Cosserat continuum

(3.19),(3.11)	$T^S_{ij,j} + e_{ikj}\, t_{k,j} + b_i = 0\,,\quad M_{ij,j} + c_i + 2\, t_i = 0$
(3.17)	$T^S_{ij} = \hat{T}^S_{ij}(\nabla^S v, 2\varphi, \nabla\omega)\,,\quad M_{ij} = \hat{M}_{ij}(\nabla^S v, 2\varphi, \nabla\omega)$ $t_i = \hat{t}_i(\nabla^S v, 2\varphi, \nabla\omega)$
(3.12)	$v_i = \hat{v}_i\,,\quad \omega_i = \hat{\omega}_i$
(3.23),(3.24)	$T^S_{\alpha n} + e_{\alpha\beta} t_\beta = s_\alpha\,,\ T^S_{nn} = s_n\,,\ M_{\alpha n} = m_\alpha\,,\ M_{nn} = m_n$

2D: Reissner plate

(5.5)	$Q_{\alpha,\alpha} + q = 0\,,\quad M_{\alpha\beta,\beta} + c_\alpha + e_{\alpha\beta} Q_\beta = 0$
(5.8)	$Q_\alpha = \hat{Q}_\alpha(\varphi, \nabla\omega),\quad M_{\alpha\beta} = \hat{M}_{\alpha\beta}(\varphi, \nabla\omega)$
(5.9)	$v_3 = \hat{v}_3\ ,\quad \omega_\alpha = \hat{\omega}_\alpha$
(5.10)	$Q_n = s_3\,,\quad M_{\alpha n} = m_\alpha$

1D: Timoshenko beam

(6.5)	$Q'_\alpha + q_\alpha = 0\,,\ \ M'_\alpha + c_\alpha - e_{\alpha\beta} Q_\beta = 0$
(6.8)	$Q_\alpha = \hat{Q}_\alpha(\varphi, \omega')\,,\ \ M_\alpha = \hat{M}_\alpha(\varphi, \omega')$
(6.9)	$v_3(l) = v_{3l}\,,\ v_3(0) = v_{30}\,,\ \ \omega_\alpha(l) = \omega_{\alpha l}\,,\ \omega_\alpha(0) = \omega_{\alpha 0}$
(6.10),(6.11)	$Q_\alpha(l) = P_{\alpha l}\,,\ Q_\alpha(0) = -P_{\alpha 0}\,,\ \ M_\alpha(l) = C_{\alpha l}\,,\ M_\alpha(0) = -C_{\alpha 0}$

* * *

3D: Constrained Cosserat continuum

(4.6)	$T^S_{ij,j} + b_i - \frac{1}{2}\, e_{ikj}\,(M_{kh,hj} + c_{k,j}) = 0$
(4.4)	$T^S_{ij} = \hat{T}^S_{ij}(\nabla^S v, \frac{1}{2}\mathrm{curl}\, v)\,,\quad M_{ij} = \hat{M}_{ij}(\nabla^S v, \frac{1}{2}\mathrm{curl}\, v)$
(4.11)	$v_\alpha = \hat{v}_\alpha\,,\quad v_n = \hat{v}_n\ ,\quad v_{\alpha,n} = \hat{v}_{\alpha,n}$
(4.12),(4.13)	$T^S_{\alpha n} + \frac{1}{2}\, e_{\alpha\beta}(M_{nn,\beta} - M_{\beta h,h}) = s_\alpha + \frac{1}{2}\, e_{\alpha\beta}(c_\beta + m_{n,\beta})\,,$ $T^S_{nn} = s_n\,,\quad M_{\alpha n} = m_\alpha$

2D: Kirchhoff-Love plate

(5.19)	$M^{*S}_{\alpha\beta,\alpha\beta} + c^*_{\alpha,\alpha} - q = 0$
(5.17)	$M^{*S}_{\alpha\beta} = \hat{M}^{*S}_{\alpha\beta}(\nabla\nabla v_3)$
(5.22)	$v_3 = \hat{v}_3\,,\ \ v_{3,n} = \hat{v}_{3,n}\,,$
(5.23)	$M^{*S}_{nn,n} + 2M^{*S}_{n\tau,\tau} = m^*_{\tau,\tau} - s_3 - c^*_n\,,\ \ M^{*S}_{nn} = m^*_n$

1D: Euler-Bernoulli beam

(6.17)	$M''_\alpha + c'_\alpha + e_{\alpha\beta} q_\beta\ =\ 0$
(6.15)	$M_\alpha = \hat{M}_\alpha(\kappa)$
(6.20)	$v_\alpha(l) = v_{\alpha l}\,,\ v_\alpha(0) = v_{\alpha 0}\,,\ \ v'_\alpha(l) = v'_{\alpha l}\,,\ \ v'_\alpha(0) = v'_{\alpha 0}$
(6.18),(6.19)	$M'_\alpha(l) + c_\alpha(l) = e_{\alpha\beta} P_{\beta l}\,,\quad M'_\alpha(0) + c_\alpha(0) = -e_{\alpha\beta} P_{\beta 0}\,,$ $M_\alpha(l) = C_{\alpha l}\,,\quad M_\alpha(0) = -C_{\alpha 0}$

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