

On the method of virtual power in the mechanics of non-classical continua

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These Notes cover four of the six lectures delivered at the Course. Their subject has been developed in the forthcoming paper (Del Piero 2013a). The two remaining lectures held in the Course were a concise presentation of the contents of the forthcoming lecture notes (Del Piero 2013b).

1 Classical mechanics and classical continua

The name *classical mechanics* generally denotes the traditional branch of mechanics, as opposed to special disciplines, such as quantum mechanics, relativistic mechanics, or statistical mechanics. *Continuum mechanics* is the branch of classical mechanics which deals with continuous deformable bodies. Its most traditional object of study, the *classical continuum*, is a continuum for which the external action consists of two systems of forces, the *body forces* and the *surface tractions*. Larger varieties of external actions define classes of non-classical continua.

Classical mechanics is based on Newton's three laws of motion. Classical continua are governed by Euler's balance laws of linear and angular momentum, which are direct consequences of Newton's first two laws.¹ The absence of the third law may look strange. In fact, as we shall see, this absence is a peculiarity of classical continua. In non-classical continua, the third law recovers its fundamental role. This is a point which is neither self-evident nor generally acknowledged, and this is one of the reasons that make interesting the study of non-classical continua.

¹(Truesdell & Toupin 1960), Sect. 196.

1.1 The traditional approach

A classical continuum is defined by two primitive elements:

- (i) a continuous body Ω ,
- (ii) a system (b, s) of external actions.

The body is identified with the region of the three-dimensional Euclidean point space \mathcal{E} occupied in a reference configuration. In the present Notes, to avoid the machinery required for changes of configurations, the current configuration is systematically taken as the reference configuration. Moreover, each point x is identified with the position vector $x - o$, with the origin o chosen once for all.

The vector $b(x)$ is the *body force* at the interior point x of Ω , and for every subregion Π of Ω the vector $s(x, \partial\Pi)$ is the *surface traction* at $x \in \partial\Pi$. Both b and s are supposed to be integrable in the respective domains. They are subject to Euler's *balance laws of linear and angular momentum*

$$\begin{aligned} \int_{\Pi} b(x) dV + \int_{\partial\Pi} s(x, \partial\Pi) dA &= 0, \\ \int_{\Pi} x \times b(x) dV + \int_{\partial\Pi} x \times s(x, \partial\Pi) dA &= 0, \end{aligned} \tag{1}$$

which are assumed to hold for every subregion Π of Ω .²

The first balance law has three important consequences on the structure of the contact actions:

- (i) (the action-reaction law) *If Π_1 and Π_2 are subregions of Ω with disjoint interiors and if x_o belongs to the boundary of both, then*

$$s(x_o, \partial\Pi_1) = -s(x_o, \partial\Pi_2), \tag{2}$$

- (ii) (the dependence on the normal) *The surface traction $s(x_o, \partial\Pi)$ depends on $\partial\Pi$ only through the exterior unit normal n to $\partial\Pi$ at x_o*

$$s(x_o, \partial\Pi) = s(x_o, n), \tag{3}$$

- (iii) (the existence of the stress tensor) *The function $s(x_o, \cdot)$ is linear. That is, there is a second-order tensor T such that*

$$s(x_o, n) = T(x_o) n. \tag{4}$$

²Inertia forces are included in the body forces, see (Noll 1963), Sect. 7.

Proofs of these statements are sketched in the next Subsection. Note that, by (3), the action-reaction law (2) takes the form

$$s(x_o, n) = -s(x_o, -n). \quad (5)$$

Substituting (4) into the first balance equation and using the Gauss-Green formula, one has

$$\int_{\Pi} b(x) dV = - \int_{\partial\Pi} T(x) n dV = - \int_{\Pi} \operatorname{div} T(x) dV, \quad (6)$$

and from the arbitrariness of Π the local equation

$$\operatorname{div} T(x) + b(x) = 0 \quad (7)$$

follows. Recalling that the cross product $a \times b$ is twice the opposite of the vector associated with the skew-symmetric part of the tensor $a \otimes b$, the second balance equation $(1)_2$ imposes the symmetry of the tensor

$$\int_{\Pi} x \otimes b(x) dV + \int_{\partial\Pi} x \otimes T(x) n dA.$$

Using again the Gauss-Green formula, this tensor takes the form

$$\int_{\Pi} (x \otimes (b(x) + \operatorname{div} T(x)) + T^T(x)) dV. \quad (8)$$

Then, by (7) and the arbitrariness of Π , condition $(1)_2$ reduces to the requirement of the symmetry of $T(x)$

$$T(x) = T^T(x). \quad (9)$$

Equations (7) and (9) are the *local forms* of the balance equations (1). Together with constitutive equations, initial conditions, and boundary conditions, they constitute the differential, or *strong*, formulation of the *problem of motion*. The *equilibrium problem* is the special case in which inertia forces are absent, and all time-dependent variables are neglected.

1.2 Sketch of the proofs

For the purpose of the present Notes, it is useful to give an idea of the proofs of the statements (i) to (iii) of the previous Subsection. The proofs given below are informal. Rigorous proofs would require precise definitions of the objects involved, for example, *regions* and *surfaces*, and a precise statement

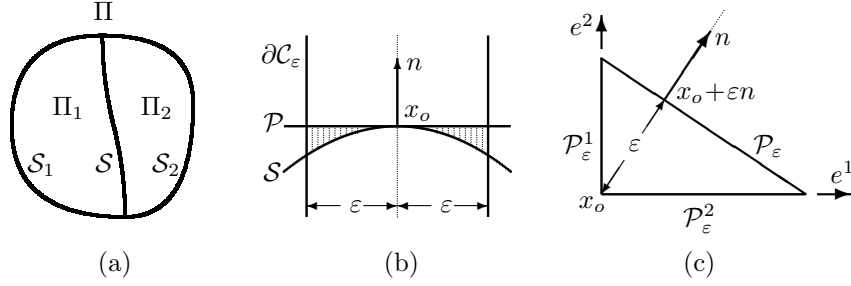


Figure 1. Two-dimensional sketches for: the decomposition of a region Π into the union of regions Π_1 and Π_2 with disjoint interiors (a), the region Π_ε (shaded area) used in the proof of Noll's theorem on the dependence on the normal (b), Cauchy's tetrahedron theorem (c).

of the regularity conditions which legitimate the use of the Gauss-Green formula.³

All proofs require the following assumption of *local dependence* of s on $\partial\Pi$: for every $x \in \partial\Pi$ there is an $\varepsilon > 0$ such that

$$s(x, \partial\Pi) = s(x, \partial\Pi \cap B_\varepsilon(x)), \quad (10)$$

where $B_\varepsilon(x)$ is the ball of radius $\varepsilon > 0$ centered at x .⁴

*Proof of (i).*⁵ Let Π be the disjoint union of two subregions Π_1 and Π_2 , as shown in Fig. 1a. By the additivity of the volume integral,

$$\int_{\Pi} b(x) dV = \int_{\Pi_1} b(x) dV + \int_{\Pi_2} b(x) dV, \quad (11)$$

and by the first balance equation written for Π, Π_1 , and Π_2 ,

$$\int_{\partial\Pi} s(x, \partial\Pi) dA = \int_{\partial\Pi_1} s(x, \partial\Pi_1) dA + \int_{\partial\Pi_2} s(x, \partial\Pi_2) dA. \quad (12)$$

To within sets of area zero, the three surfaces are the disjoint unions

$$\partial\Pi = \mathcal{S}_1 \cup \mathcal{S}_2, \quad \partial\Pi_1 = \mathcal{S}_1 \cup \mathcal{S}, \quad \partial\Pi_2 = \mathcal{S}_2 \cup \mathcal{S}, \quad (13)$$

³All this falls into the domain of *geometric measure theory*. The interested reader is addressed to the books (Vol'pert & Hudjaev 1985) and (Capriz 1989), and to the papers (Ziemer 1983), (Šilhavý 1985, 1991), (Gurtin et al. 1986), (Capriz & Virga 1990), (Schuricht 2007), (Chen et al. 2009).

⁴This is the *principle of local action*, (Truesdell & Noll 1965).

⁵(Noll 1959), Theorem III.

with

$$\mathcal{S} = \partial\Pi_1 \setminus \partial\Pi = \partial\Pi_2 \setminus \partial\Pi, \quad \mathcal{S}_1 = \partial\Pi \cap \partial\Pi_1, \quad \mathcal{S}_2 = \partial\Pi \cap \partial\Pi_2. \quad (14)$$

By the additivity of the surface integral, from (12) and (13) we get

$$\begin{aligned} \int_{\mathcal{S}_1} s(x, \partial\Pi) dA + \int_{\mathcal{S}_2} s(x, \partial\Pi) dA &= \int_{\mathcal{S}_1} s(x, \partial\Pi_1) dA \\ &+ \int_{\mathcal{S}} s(x, \partial\Pi_1) dA + \int_{\mathcal{S}_2} s(x, \partial\Pi_2) dA + \int_{\mathcal{S}} s(x, \partial\Pi_2) dA. \end{aligned} \quad (15)$$

By assumption (10), the two integrals over \mathcal{S}_1 cancel. The same do the two integrals over \mathcal{S}_2 . It remains

$$\int_{\mathcal{S}} (s(x, \partial\Pi_1) + s(x, \partial\Pi_2)) dA = 0.$$

The surface $\partial\Pi$ has a natural orientation, with the interior on the side of Π and the exterior on the side of $\Omega \setminus \Pi$. The same holds for $\partial\Pi_1$ and $\partial\Pi_2$. When, as in (13), these regions are split into subsurfaces $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}$, the subsurfaces inherit the corresponding orientations. That is, \mathcal{S}_1 is oriented as $\partial\Pi$ in (13)₁ and as $\partial\Pi_1$ in (13)₂, and the two orientations coincide because the interiors of Π and Π_1 are on the same side of the surface.

The same holds for \mathcal{S}_2 . On the contrary, \mathcal{S} is oriented as $\partial\Pi_1$ in (13)₂ and as $\partial\Pi_2$ in (13)₃, and the two orientations are opposite. Let us denote by $\vec{\mathcal{S}}$ the surface \mathcal{S} oriented with the interior on the side of Π_1 , and by $\overleftarrow{\mathcal{S}}$ the same surface with the interior on the side of Π_2 . Then, by (10),

$$\int_{\mathcal{S}} (s(x, \vec{\mathcal{S}}) + s(x, \overleftarrow{\mathcal{S}})) dA = 0. \quad (16)$$

Take a point x_o in \mathcal{S} , and let \mathcal{S}_ε be the intersection of \mathcal{S} with $B_\varepsilon(x_o)$. If

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \frac{1}{A(\mathcal{S}_\varepsilon)} \int_{\mathcal{S}_\varepsilon} s(x, \vec{\mathcal{S}}) dA &= s(x_o, \vec{\mathcal{S}}), \\ \lim_{\varepsilon \rightarrow 0} \frac{1}{A(\mathcal{S}_\varepsilon)} \int_{\mathcal{S}_\varepsilon} s(x, \overleftarrow{\mathcal{S}}) dA &= s(x_o, \overleftarrow{\mathcal{S}}), \end{aligned} \quad (17)$$

then, by (16),

$$s(x_o, \vec{\mathcal{S}}) = -s(x_o, \overleftarrow{\mathcal{S}}). \quad (18)$$

This shows that s depends on the orientation of the surface, in the way stated in (2). \square

*Proof of (ii).*⁶ Let x_o be an interior point of Ω , and let \mathcal{S} be a smooth surface with unit normal n at x_o . Moreover, let \mathcal{P} be the tangent plane to \mathcal{S} at x_o , and let \mathcal{C}_ε be the cylinder of radius ε , whose axis is the line parallel to n from x_o . Let, further, Π_ε be the part of \mathcal{C}_ε included between \mathcal{S} and \mathcal{P} . As shown in Fig. 1b, the boundary $\partial\Pi_\varepsilon$ is the disjoint union

$$\partial\Pi_\varepsilon = (\partial\Pi_\varepsilon \cap \mathcal{S}) \cup (\partial\Pi_\varepsilon \cap \mathcal{P}) \cup (\partial\Pi_\varepsilon \cap \partial\mathcal{C}_\varepsilon). \quad (19)$$

Then, by the balance equation (1)₁,

$$\int_{\Pi_\varepsilon} b(x) dV + \int_{\partial\Pi_\varepsilon \cap \mathcal{S}} s(x, \mathcal{S}) dA + \int_{\partial\Pi_\varepsilon \cap \mathcal{P}} s(x, \mathcal{P}) dA + \int_{\partial\Pi_\varepsilon \cap \partial\mathcal{C}_\varepsilon} s(x, \partial\mathcal{C}_\varepsilon) dA = 0. \quad (20)$$

The areas of the surfaces in $\partial\Pi_\varepsilon$ and the volume of Π_ε are

$$\begin{aligned} A(\partial\Pi_\varepsilon \cap \mathcal{P}) &= \pi\varepsilon^2, & A(\partial\Pi_\varepsilon \cap \mathcal{S}) &= \pi\varepsilon^2 + o(\varepsilon^2), \\ A(\partial\Pi_\varepsilon \cap \partial\mathcal{C}_\varepsilon) &= o(\varepsilon^2), & V(\Pi_\varepsilon) &= o(\varepsilon^3), \end{aligned} \quad (21)$$

respectively. If

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \frac{1}{A(\partial\Pi_\varepsilon \cap \mathcal{P})} \int_{\partial\Pi_\varepsilon \cap \mathcal{P}} s(x, \mathcal{P}) dA &= s(x_o, \mathcal{P}), \\ \lim_{\varepsilon \rightarrow 0} \frac{1}{A(\partial\Pi_\varepsilon \cap \mathcal{S})} \int_{\partial\Pi_\varepsilon \cap \mathcal{S}} s(x, \mathcal{S}) dA &= s(x_o, \mathcal{S}), \\ \lim_{\varepsilon \rightarrow 0} \frac{1}{A(\partial\Pi_\varepsilon \cap \partial\mathcal{C}_\varepsilon)} \int_{\partial\Pi_\varepsilon \cap \partial\mathcal{C}_\varepsilon} s(x, \partial\mathcal{C}_\varepsilon) dA &= 0, \end{aligned} \quad (22)$$

from (20) divided by $\pi\varepsilon^2$, in the limit for $\varepsilon \rightarrow 0$ it follows that

$$s(x_o, \mathcal{S}) + s(x_o, \mathcal{P}) = 0.$$

Denoting by $\vec{\mathcal{P}}, \vec{\mathcal{S}}$ the surfaces \mathcal{P}, \mathcal{S} oriented with the exterior normal n , this equation becomes $s(x_o, \vec{\mathcal{S}}) + s(x_o, \vec{\mathcal{P}}) = 0$. That is, by (18),

$$s(x_o, \vec{\mathcal{S}}) = s(x_o, \vec{\mathcal{P}}).$$

This equality holds for every oriented surface $\vec{\mathcal{S}}$ with exterior normal n at x_o . Therefore, $s(x_o, \vec{\mathcal{S}})$ is the same for all such surfaces, and (3) follows. \square

*Proof of (iii).*⁷ Consider an interior point x_o of Ω , an orthonormal basis e^i , and a unit vector n with components

$$n_i = n \cdot e^i. \quad (23)$$

⁶(Noll 1959), Theorem IV.

⁷(Cauchy 1823).

There is no loss in generality in choosing the orientations of the e^i such that all n_i are positive.

For fixed $\varepsilon > 0$, let Π_ε be the region between the planes through x_o with normals e^i , and the plane through $x_o + \varepsilon n$ with normal n . As shown in the simplified two-dimensional representation of Fig. 1c, Π_ε is the tetrahedron with vertex x_o and height ε , whose basis \mathcal{P}_ε has exterior unit normal n , while the three faces $\mathcal{P}_\varepsilon^i$ have exterior unit normals $-e^i$. Then, by (3) and the balance equation (1)₁,

$$\int_{\Pi_\varepsilon} b(x) dV + \int_{\mathcal{P}_\varepsilon} s(x, n) dA + \sum_{i=1}^3 \int_{\mathcal{P}_\varepsilon^i} s(x, -e^i) dA = 0. \quad (24)$$

If $A(\mathcal{P}_\varepsilon)$ is the area of the basis, the areas of the faces and the volume of Π_ε are

$$A(\mathcal{P}_\varepsilon^i) = A(\mathcal{P}_\varepsilon) n_i, \quad V(\Pi_\varepsilon) = \frac{1}{3} \varepsilon A(\mathcal{P}_\varepsilon), \quad (25)$$

respectively. If

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \frac{1}{A(\mathcal{P}_\varepsilon^i)} \int_{\mathcal{P}_\varepsilon^i} s(x, -e^i) dA &= s(x_o, -e^i), \\ \lim_{\varepsilon \rightarrow 0} \frac{1}{A(\mathcal{P}_\varepsilon)} \int_{\mathcal{P}_\varepsilon} s(x, n) dA &= s(x_o, n), \end{aligned} \quad (26)$$

from (24) divided by $A(\mathcal{P}_\varepsilon)$, in the limit for $\varepsilon \rightarrow 0$ one gets

$$s(x_o, n) + \sum_{i=1}^3 s(x_o, -e^i) n_i = 0. \quad (27)$$

That is, by (5),

$$s(x_o, n) = \sum_{i=1}^3 s(x_o, e^i) (n \cdot e^i) = \left(\sum_{i=1}^3 s(x_o, e^i) \otimes e^i \right) n. \quad (28)$$

This is the desired relation (4), with

$$T(x) = \sum_{i=1}^3 s(x, e^i) \otimes e^i. \quad \square \quad (29)$$

Equations (17), (22), and (26) are the delicate parts of the proofs. In (17), (22)₁, (22)₂, and (26)₁, the limit is made over a family of subsurfaces of a fixed surface, and all members of the family contain the point x_o . Then

the corresponding equalities hold for almost every x_o on the surface.⁸ On the contrary, the limits in (22)₃ and (26)₂ are made over families of pairwise disjoint surfaces \mathcal{P}_ε , none of which contains x_o . In this case, the extra assumption of the continuity of $s(\cdot, n)$ at x_o would be required. However, in (Gurtin et al. 1968), using a mollifying procedure, this requirement has been replaced by the milder assumption of $s(\cdot, n)$ integrable in Ω .

All proofs given above refer to a vector field $s(x, n)$. Similar statements hold for scalar fields and for tensor fields of any order. For a scalar field σ , it can be proved that there exists a vector field Σ such that⁹

$$\sigma(x, n) = \Sigma(x) \cdot n, \quad \sigma = \Sigma_i n_i, \quad (30)$$

and for a second-order tensor field S it can be proved that there exists a third-order tensor field \mathbb{T} such that

$$S(x, n) = \mathbb{T}(x) n, \quad S_{ij} = \mathbb{T}_{ijk} n_k. \quad (31)$$

1.3 The indifference of power

In continuum mechanics, the balance equations (1) are taken as postulates. This view, which works perfectly for classical continua, cannot be easily generalized to non-classical continua. This justifies a re-consideration of the position of the balance equations in the theory.

A first step in this direction was to show that the balance equations can be deduced from the more fundamental *principle of indifference of the external power* under changes of observer.¹⁰ To state this principle we need to add a third object to the two primitive elements mentioned above as essential to define a classical continuum. This is the set of the *virtual displacements*.

Virtual displacements are the initial velocities in a possible motion from the current configuration. With every virtual displacement v is associated the *external power*

$$P_{ext}(\Pi, v) = \int_{\Pi} b(x) \cdot v(x) dV + \int_{\partial\Pi} s(x, \partial\Pi) \cdot v(x) dA, \quad (32)$$

spent in the portion Π of the body by the system (b, s) of external actions. Since the motion need not be a *real* motion, the velocities are called *virtual*

⁸That is, to within a set of zero volume. This is by the Lebesgue-Besicovitch theorem, see e.g. (Evans & Garipey 1992), Section 1.7.

⁹With summation over repeated indices. For a proof of (31) see (Del Piero 2009).

¹⁰(Noll 1963).

velocities or *virtual displacements*, and the just defined external power is called a *virtual power*.

Because they describe infinitesimal changes of the current configuration, the virtual displacements are considered as infinitesimal. A *change of observer* is a transformation

$$v(x) \mapsto v(x) + c + Wx, \quad (33)$$

with c an arbitrary vector and W an arbitrary skew-symmetric tensor. This is the most general infinitesimal transformation which leaves unaltered the mutual distances between all pairs of points of Ω . It seems reasonable to assume that the power associated with such transformations is zero. In view of the linearity of $P_{ext}(\Pi, \cdot)$, the indifference of the external power to all transformations of the type (33) is expressed by the conditions

$$P_{ext}(\Pi, c) = 0, \quad P_{ext}(\Pi, Wx) = 0. \quad (34)$$

They are called the condition of *translational* and of *rotational* indifference, respectively. The first condition gives directly the balance equation (1)₁. The second gives the equation

$$W \cdot \left(\int_{\Pi} b(x) \otimes x \, dV + \int_{\partial\Pi} s(x, \partial\Pi) \otimes x \, dA \right),$$

from which (1)₂ follows. Because the balance equations (1) are consequences of the indifference of power, the latter can be taken as a basic postulate of classical continuum mechanics in place of Euler's balance laws.

1.4 The method of virtual power

With the use of the local balance equations (7), (9), and of the Gauss-Green formula, the external power can be transformed into a volume integral

$$\begin{aligned} & \int_{\Pi} b(x) \cdot v(x) \, dV + \int_{\partial\Pi} s(x, \partial\Pi) \cdot v(x) \, dA \\ &= - \int_{\Pi} \operatorname{div} T(x) \cdot v(x) \, dV + \int_{\partial\Pi} T(x) n \cdot v(x) \, dA = \int_{\Pi} T(x) \cdot \nabla v(x) \, dV, \end{aligned}$$

called the *internal power* and denoted by $P_{int}(\Pi, v)$. Recalling that, for any symmetric tensor T ,

$$T(x) \cdot \nabla v(x) = T^S(x) \cdot \nabla v(x) = T(x) \cdot \nabla^S v(x), \quad (35)$$

where T^S and $\nabla^S v$ are the symmetric parts of T and ∇v , the internal power reduces to

$$P_{int}(\Pi, v) = \int_{\Pi} T(x) \cdot \nabla^S v(x) \, dV. \quad (36)$$

The integral identity

$$P_{ext}(\Pi, v) = P_{int}(\Pi, v) \quad (37)$$

is the *equation of virtual power*.

The problems met in formulating balance equations for non-classical continua induced several authors to reverse the traditional approach described in Subsection 1.1, by assuming the equation of virtual power as a postulate, and deducing from it the balance equations.

This is the *method of virtual power*.¹¹ In this method, the balance equations (1) are deduced using the Gauss-Green formula and the arbitrariness of Π . Notice that, since the existence of the Cauchy stress is now assumed, the relation $s = Tn$ follows directly from the equation of virtual power, without the use of the tetrahedron theorem.

For a classical continuum, this method is equivalent to the traditional approach. Some problems arise for non-classical continua. Indeed, due to the presence of supplementary internal actions, there are many possible expressions of the internal power from which Euler's laws can be deduced. Each of them produces its own extra balance equations for the supplementary actions. Therefore, a curious situation is created: a postulate, the principle of virtual power, is required to be compatible with Euler's balance laws (1) but, at the same time, other balance laws are deduced from the postulate itself. We will be back to this point.

1.5 The variational approach

The variational approach consists in deducing the equation of virtual power from a minimum principle of an energy functional. For a classical continuum the energy is the sum of two terms, the strain energy and the energy of the loads. The energy of the loads is the opposite of the external power (32), and the strain energy is a function of the deformation u , depending on the specific material which constitutes the body. For an *elastic material* the strain energy has a volume density w , which is a function of the current value of ∇u . The total energy has the form

$$E(u) = \int_{\Omega} (w(\nabla u(x)) - b(x) \cdot u(x)) dV - \int_{\partial\Omega} s(x) \cdot u(x) dA.$$

Denoting by v a perturbation of u , the first variation of E is

$$\delta E(u, v) = \int_{\Omega} (\nabla w(\nabla u(x)) \cdot \nabla v(x) - b(x) \cdot v(x)) dV - \int_{\partial\Omega} s(x) \cdot v(x) dA.$$

¹¹(Germain 1973a, 1973b).

After setting

$$T(x) = \nabla w(\nabla u(x)),$$

we see that the Euler-Lagrange equation $\delta E(u, v) = 0$ coincides with the equation of virtual power (37).

Therefore, the variational approach is not substantially different from the method of virtual power. The only difference is that the variational approach requires the specification of the functional dependence of the strain energy on the deformation, that is, the choice of a specific material, since the very beginning. In the method of virtual power this choice can be postponed, since the equation of virtual power is independent of the constitutive equations.

For an elastic material, the stress is a function of the gradient of the energy density. For other materials, for example, elastic-plastic, visco-elastic, or visco-plastic materials, the total energy contains some dissipative parts.¹² In this case, the constitutive equations become more complicated. For example, they may depend on the past history of the deformation or on supplementary state variables. These subjects will not be treated in the present Notes.

1.6 Bounded Cauchy fluxes

There is an alternative approach to classical continuum mechanics, which emerged over the years from the work of several authors.¹³ This is the approach based on the concept of Cauchy flux.¹⁴ Till now, this approach did not receive adequate attention, and did not reach a fully satisfactory settling.¹⁵

A *Cauchy flux* is a function Q from the interior surfaces of Ω into the vectors, additive on disjoint surfaces belonging to the boundary of the same subregion Π of Ω

$$\mathcal{S}_1, \mathcal{S}_2 \subset \partial\Pi, \quad \mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset \quad \Rightarrow \quad Q(\mathcal{S}_1 \cup \mathcal{S}_2) = Q(\mathcal{S}_1) + Q(\mathcal{S}_2). \quad (38)$$

Here \mathcal{S}_1 and \mathcal{S}_2 are oriented surfaces, with the same orientation as the natural orientation of $\partial\Pi$. A Cauchy flux is *skew-symmetric* if

$$Q(\vec{\mathcal{S}}) = -Q(\overleftarrow{\mathcal{S}}). \quad (39)$$

¹²(Halphen & Nguyen 1975), (Mielke 2011).

¹³(Stippes 1971), (Noll 1973), (Gurtin & Martins 1976), (Ziemer 1983), (Šilhavý 1985, 1991).

¹⁴(Gurtin & Martins 1976).

¹⁵For some steps in this direction see (Del Piero 2009, 2013a).

A skew-symmetric Cauchy flux has the additivity property

$$Q(\partial(\Pi_1 \cup \Pi_2)) = Q(\partial\Pi_1) + Q(\partial\Pi_2), \quad (40)$$

for all regions Π_1, Π_2 of Ω with disjoint interiors. Indeed, let $\mathcal{S}_1, \mathcal{S}_2$ and \mathcal{S} be as in (14), with $\Pi = \Pi_1 \cup \Pi_2$, and with $\mathcal{S}_1, \bar{\mathcal{S}}$ oriented as $\partial\Pi_1$ and $\mathcal{S}_2, \bar{\mathcal{S}}$ oriented as $\partial\Pi_2$. By (13) and (38),

$$\begin{aligned} Q(\partial(\Pi_1 \cup \Pi_2)) &= Q(\mathcal{S}_1) + Q(\mathcal{S}_2), \\ Q(\partial\Pi_1) &= Q(\mathcal{S}_1) + Q(\bar{\mathcal{S}}), \quad Q(\partial\Pi_2) = Q(\mathcal{S}_2) + Q(\bar{\mathcal{S}}), \end{aligned} \quad (41)$$

and (40) follows from (39).

The restriction of Q to the boundaries $\partial\Pi$ can be regarded as a function F mapping the subregions Π into the vectors

$$F(\Pi) = -Q(\partial\Pi). \quad (42)$$

By (40), F is additive on disjoint regions if and only if Q is skew-symmetric. Assume that for every subregion Π of Ω there is a non-negative, scalar-valued function h^Π , integrable over $\partial\Pi$ and such that

$$|Q(\mathcal{S})| \leq \int_{\mathcal{S}} h^\Pi(x) dA, \quad (43)$$

for every subsurface \mathcal{S} of $\partial\Pi$. Moreover, assume that there is a non-negative, scalar-valued function h , integrable over Ω and such that

$$|F(\Pi)| \leq \int_{\Pi} h(x) dV, \quad (44)$$

for every subregion Π of Ω . Under these assumptions, it has been proved¹⁶ that Q has a surface density $s(\cdot, \partial\Pi)$

$$Q(\mathcal{S}) = \int_{\mathcal{S}} s(x, \partial\Pi) dA, \quad (45)$$

for every subsurface \mathcal{S} of $\partial\Pi$, and F has a volume density f

$$F(\Pi) = \int_{\Pi} f(x) dV, \quad (46)$$

for every subregion Π of Ω . By the definition (42), $-f$ can be interpreted as the volume density of Q .

¹⁶(Šilhavý 1985), Proposition 1 and Theorem 1.

Condition (46) implies the additivity of F on disjoint subsets, that is, the skew-symmetry of Q . A skew-symmetric Cauchy flux with surface and volume densities will be called a *bounded Cauchy flux*. For fluxes of this type, equation (42) takes the form

$$\int_{\partial\Pi} s(x, \partial\Pi) dA + \int_{\Pi} f(x) dV = 0. \quad (47)$$

This equation has the same form of the equation (1)₁ of balance of linear momentum. But *it is not* a balance equation, since it does not express the balance of any physical quantities. It is only a relation between the surface and volume densities of Q . Due to its formal resemblance with a balance equation, it will be called a *pseudobalance equation*.

Starting from this equation, the properties (3), (4) of the dependence of $s(x, \partial\Pi)$ on the normal and of the existence of the stress tensor can be proved as done in Subsection 1.2, just by replacing b with f . Thus, a distinctive property of the approach based on bounded Cauchy fluxes is that *the existence of the stress tensor is not a consequence of Euler's law* (1)₁.

For a classical continuum, using the Gauss-Green formula, the external power (32) can be transformed into the volume integral

$$\int_{\Pi} ((b(x) + \operatorname{div}T(x)) \cdot v(x) + T(x) \cdot \nabla v(x)) dV,$$

which, by the symmetry of T imposed by the rotational indifference requirement (34)₂, reduces to

$$\int_{\Pi} T(x) \cdot \nabla^S v(x) dV. \quad (48)$$

This integral coincides with the internal power (36). Thus, *the equation of virtual power for a classical continuum coincides with the pseudobalance equation for a bounded Cauchy flux*. This leads to the identification of the body force b with the volume density f .

To summarize, the alternative approach discussed in this Subsection is based on two main assumptions:

- (i) *the system of contact actions is a bounded Cauchy flux,*
- (ii) *the external power is indifferent.*

These assumptions coincide with Newton's three laws of motion plus some extra regularity requirements. Indeed, as shown in Subsection 1.3, the first two laws follow from the indifference of power, and the third law, or action-reaction law, is a property of the skew-symmetric Cauchy fluxes.

In the classical approach, the third law is a consequence of the first two. In the present approach, the three laws are independent. Thus, for the classical continuum the traditional approach seems to be more convenient. As we shall see, this advantage disappears when dealing with non-classical continua.

2 Non-classical continua

2.1 Continua with microstructure

A *continuum with microstructure* is a continuum in which the deformation acts on two length scales of different order of magnitude, macroscopic and microscopic.¹⁷ The displacement vector u describes the *macroscopic deformation*, while the microdeformation is described by a finite number of *order parameters*, also called *internal variables* or *state variables*. They can be scalars, vectors, or tensors, depending on the physical nature of the microstructure.

With u is associated the system (b, s) of external actions of the classical continuum, and with each d^α are associated a *body microforce* β^α and a *surface microtraction* σ^α , of the same tensorial nature of d^α . Denoting by v and ν^α the virtual variations of u and d^α , the *external power* is defined by

$$P_{ext}(\Pi, v, \nu^\alpha) = \int_{\Pi} (b \cdot v + \beta^\alpha \cdot \nu^\alpha) dV + \int_{\partial\Pi} (s \cdot v + \sigma^\alpha \cdot \nu^\alpha) dA, \quad (49)$$

where summation over repeated superscripts α is understood. For simplicity of notation, from here onwards the reference to the argument x is omitted.

Models for continua with microstructure can be constructed by generalizing the approaches discussed in the previous sections for classical continua. In the traditional approach, for each order parameter a *microforce balance equation* of the same form of the pseudobalance equation (47) is assumed.¹⁸ The status of such equations is not completely clear. Indeed, due to their

¹⁷The literature on this subject is very large. The starting point was the theory of elastic bodies with couple stresses of the Cosserats (1909), revived at the beginning of the 1960's in papers of (Grioli 1960), (Aero & Kuvshinskii 1960), (Toupin 1962), (Mindlin & Tiersten 1962), and others. Straight after, the study of the more general class of *micropolar continua* was started by (Mindlin 1964), (Eringen 1964), (Green 1965), and others. The surprising number of contributions produced at the end of the decade is documented in the proceedings of the IUTAM Conference (Kröner 1968) and in the lecture notes (Stojanović 1969). A history of successive developments and a broad list of applications can be found in the book (Capriz 1989).

¹⁸(Capriz 1989).

variable number and nature, they can hardly be considered as general laws of mechanics. Though I was not able to find any precise statement about the nature of microforce balance equations, I believe it reasonable to consider them as *constitutive assumptions* defining specific classes of continua.

In the approach based on the method of virtual power, a generalized expression of the internal power is assumed.¹⁹ In this case, it is not clear how much freedom is allowed in the choice of the expression of the internal power. In most expressions proposed in the literature, the *structural properties* of a continuum, dictated by the choice of the order parameters, are mixed with constitutive assumptions, which, as explained below, are of a completely different nature.

To generalize the approach introduced in Subsection 1.6, the basic assumption is that each system σ^α of surface microtractions is the surface density of a bounded Cauchy flux Q^α . Then σ^α has the property (3) of the dependence on the normal, and the linearity property

$$\sigma^\alpha(x, n) = \Sigma^\alpha(x) n,$$

which may take the form (4), (30), or (31), depending on the tensorial nature of the order parameter d^α . Moreover, each Q^α has a volume density ϕ^α , for which the counterpart of the pseudobalance equation (47)

$$\int_{\partial\Pi} \sigma^\alpha(x, \partial\Pi) dA + \int_{\Pi} \phi^\alpha(x) dV = 0, \quad (50)$$

holds. From (47) and (50), the local forms

$$\operatorname{div}T + f = 0, \quad \operatorname{div}\Sigma^\alpha + \phi^\alpha = 0, \quad (51)$$

can be deduced. Using the Gauss-Green formula, the external power transforms into the volume integral

$$P_{int}(\Pi, v, \nu^\alpha) = \int_{\Pi} ((b - f) \cdot v + T \cdot \nabla v + (\beta^\alpha - \phi^\alpha) \cdot \nu^\alpha + \Sigma^\alpha \cdot \nabla \nu^\alpha) dV, \quad (52)$$

called the *internal power*. Equating to the external power (49), the equation of virtual power

$$P_{ext}(\Pi, v, \nu^\alpha) = P_{int}(\Pi, v, \nu^\alpha) \quad (53)$$

is obtained. Just like equation (37) for classical continua, in the present approach the equation of virtual power is in fact an identity, which holds when all systems of contact actions are bounded Cauchy fluxes. Both terms

¹⁹(Germain 1973a; 1973b).

of the identity are determined by the choice of the order parameters. We say that this choice determines the *structural properties* of a continuum.

In the following, we will consider conditions for the indifference of the internal power. Indeed, with the equation of virtual power reduced to an identity, the external power is indifferent if and only if the internal power is. In all examples discussed below, the internal power has the translational indifference property

$$P_{int}(\Pi, c, 0) = 0, \quad (54)$$

whose consequence is the identification of the volume density f with the body force b

$$f = b. \quad (55)$$

In general, there is no identification of the volume densities ϕ^α with the body microforces β^α . Therefore, the internal forces²⁰

$$\zeta^\alpha = \beta^\alpha - \phi^\alpha \quad (56)$$

appear in (52). Then the internal power reduces to

$$P_{int}(\Pi, v, \nu^\alpha) = \int_{\Pi} (T \cdot \nabla v + \zeta^\alpha \cdot \nu^\alpha + \Sigma^\alpha \cdot \nabla \nu^\alpha) dV. \quad (57)$$

This is *the most general form* of the internal power for a continuum with microstructure. It consists of the products of the *internal forces* $T, \zeta^\alpha, \Sigma^\alpha$ by the *generalized deformations* $\nabla v, \nu^\alpha, \nabla \nu^\alpha$. The nature of the virtual velocities ν^α determines the *structural properties* of the continuum. Equation (57) is independent of the *constitutive equations*, which are relations between internal forces and generalized deformations, required to complete the formulation of the problem of motion. The study of such relations is out of the purposes of the present Notes.

Restrictions on the form of the internal power are expected to come from the condition of rotational indifference. They take different forms for different continua, depending on the physical nature of the order parameters. Therefore, *classes of continua* are determined by the structural properties and by the rotational indifference requirements. In particular, a *classical continuum* is a continuum with no order parameters and with the rotational indifference condition (34)₂.

Non-classical continua may have a scalar, vectorial, or tensorial microstructure. In the following Subsections, some examples of each of these classes of continua are briefly discussed.

²⁰(Capriz 1989), Sect. 8.

2.2 Continua with scalar microstructure

Let us give some examples of continua with scalar microstructure, taken from the literature. All were constructed using the method of virtual power.

In the model for damage of Frémond & Nedjar (1996), damage is attributed to microscopic motions which modify the macroscopic properties of the material. At the macroscopic scale, the effects of the microscopic motions are represented by a single scalar order parameter d , the *intensity of damage*. The assumed expressions of the external and internal power are the scalar versions of (49) and (57)

$$\begin{aligned} P_{ext}(\Pi, v, \nu) &= \int_{\Pi} (b \cdot v + \beta \nu) dV + \int_{\partial\Pi} (s \cdot v + \sigma \nu) dA, \\ P_{int}(\Pi, v, \nu) &= \int_{\Pi} (T \cdot \nabla v + \zeta \nu + \Sigma \cdot \nabla \nu) dV, \end{aligned} \quad (58)$$

with ν the virtual variation of d , β the scalar body microforce, σ the scalar surface microtraction, ζ the damage internal force, and Σ the damage flux vector. Using the Gauss-Green formula, the internal power takes the form

$$\int_{\Pi} (-\operatorname{div} T \cdot v + (\zeta - \operatorname{div} \Sigma) \nu) dV + \int_{\partial\Pi} (T n \cdot v + (\Sigma \cdot n) \nu) dA,$$

and equating to the external power, the balance equation (7) of linear momentum and the microforce balance equation

$$\operatorname{div} \Sigma + \beta = \zeta, \quad (59)$$

are obtained, together with the relations

$$s = T n, \quad \sigma = \Sigma \cdot n. \quad (60)$$

A scalar microstructure is also assumed in the scalar theories of strain-gradient plasticity.²¹ In it, the gradient ∇v is decomposed into the sum of an elastic and a plastic part²²

$$\nabla v = \nabla^e v + \nabla^p v, \quad (61)$$

and for the plastic part a single scalar measure ν is assumed, by setting

$$\nabla^p v = \nu M^p,$$

²¹(Fleck & Hutchinson 2001), (Fleck & Willis 2009a), (Niordson & Hutchinson 2011).

²²This decomposition originates from the multiplicative decomposition $F = F^e F^p$ of the deformation gradient, and holds for infinitesimal displacement gradients when, as done here, the current configuration is taken as the reference configuration.

with M^p a constant second-order tensor. The external power is as in (58)₁, while for the internal power the form

$$\begin{aligned} P_{int}(\Pi, v, \nu) &= \int_{\Pi} (T \cdot \nabla^e v + \zeta^\# \nu + \Sigma \cdot \nabla \nu) dV \\ &= \int_{\Pi} (T \cdot \nabla v + (\zeta^\# - T \cdot M^p) \nu + \Sigma \cdot \nabla \nu) dV \end{aligned} \quad (62)$$

is assumed. This is a special case of (58)₂, with $\zeta = \zeta^\# - T \cdot M^p$.

An example involving several scalar order parameters ν^α is *crystal plasticity*.²³ In it, the plastic strain rate is supposed to be a sum of dyads

$$\nabla^p v = \nu^\alpha e^\alpha \otimes m^\alpha, \quad (63)$$

where e^α are slip directions, one for each of a finite number of planes with unit normals m^α , and ν^α are virtual slip intensities, or virtual *microshear rates*. The dyad $e^\alpha \otimes m^\alpha$ is the α -th *Schmid tensor*. The external power is the scalar version of (49), and for the internal power the form

$$\begin{aligned} P_{int}(\Pi, v, \nu^\alpha) &= \int_{\Pi} (T \cdot \nabla^e v + \zeta^{\alpha\#} \nu^\alpha + \Sigma^\alpha \cdot \nabla \nu^\alpha) dV \\ &= \int_{\Pi} (T \cdot \nabla v + (\zeta^{\alpha\#} - T \cdot e^\alpha \otimes m^\alpha) \nu^\alpha + \Sigma^\alpha \cdot \nabla \nu^\alpha) dV \end{aligned} \quad (64)$$

is assumed. This is also a special case of (58)₂, with $\zeta^\alpha = \zeta^{\alpha\#} - T \cdot e^\alpha \otimes m^\alpha$. The scalar $T \cdot e^\alpha \otimes m^\alpha$ is the *resolved shear stress*. From the divergence theorem, the balance equation (51)₁ and the microforce balance equations

$$\operatorname{div} \Sigma^\alpha + \beta^\alpha = \zeta^{\alpha\#} - T \cdot e^\alpha \otimes m^\alpha \quad (65)$$

follow.

2.3 Continua with vectorial microstructure

In some vectorial microstructures, the order parameters d^α are vectors representing material directions, for example, the orientation of the crystalline lattice or the directions of crystal defects. If, as it often occurs, the d^α are supposed to be inextensible, they are called *directors*. The continua with a microstructure defined by directors are called *micropolar continua*.²⁴

For this type of continua we give a description based on bounded Cauchy fluxes, starting from an external power of the form (49). The pseudobalance

²³(Hill 1966), (Rice 1971), (Gurtin 2003b), (Gurtin et al. 2010) Sect. 105.

²⁴(Eringen 1966). For a more recent review see (Pabst 2005).

equations have the form (47) and (50), their local forms are as in (51), and the internal power is as in (57), with ζ^α and ν^α vectors and Σ^α second-order tensors.²⁵

Typical of micropolar continua is the form taken by rotational indifference. Since the directors may change their orientation when the body deforms, the external power is required to be invariant under simultaneous rigid rotations of the body and of the directors

$$P_{int}(\Pi, Wx, Wd^\alpha) = 0. \quad (66)$$

Then, from (57),

$$\begin{aligned} 0 &= \int_{\Pi} (T \cdot W + \zeta^\alpha \cdot Wd^\alpha + \Sigma^\alpha \cdot W\nabla d^\alpha) dV \\ &= W \cdot \int_{\Pi} (T + \zeta^\alpha \otimes d^\alpha + \Sigma^\alpha \nabla^T d^\alpha) dV, \end{aligned} \quad (67)$$

with $\nabla^T d^\alpha$ the transpose of ∇d^α . From the arbitrariness of W and Π it follows that the tensor T is not symmetric, and that its skew-symmetric part is

$$T^W = -(\zeta^\alpha \otimes d^\alpha + \Sigma^\alpha \nabla^T d^\alpha)^W. \quad (68)$$

Then the internal power reduces to

$$\begin{aligned} P_{int}(\Pi, v, \nu^\alpha) &= \int_{\Pi} (T \cdot \nabla^S v - (\zeta^\alpha \otimes d^\alpha + \Sigma^\alpha \nabla^T d^\alpha) \cdot \nabla^W v \\ &\quad + \zeta^\alpha \cdot \nu^\alpha + \Sigma^\alpha \cdot \nabla \nu^\alpha) dV \\ &= \int_{\Pi} (T \cdot \nabla^S v + \zeta^\alpha \cdot (\nu^\alpha - \nabla^W v d^\alpha) \\ &\quad + \Sigma^\alpha \cdot (\nabla \nu^\alpha - \nabla^W v \nabla d^\alpha)) dV \\ &= \int_{\Pi} (T \cdot \nabla^S v + \zeta^\alpha \cdot \psi^\alpha + \Sigma^\alpha \cdot \Psi^\alpha) dV, \end{aligned} \quad (69)$$

where

$$\nabla^S v, \quad \psi^\alpha = \nu^\alpha - \nabla^W v d^\alpha, \quad \Psi^\alpha = \nabla \nu^\alpha - \nabla^W v \nabla d^\alpha, \quad (70)$$

are the generalized deformations corresponding to the internal forces T , ζ^α , and Σ^α , respectively. The vectors ψ^α are the relative rotations between the directors d^α and the corresponding directions in the deformed body.

²⁵Following the traditional approach, in the model for nematic elastomers of (Anderson et al. 1999) a microforce balance equation of the form (59), which the authors call *orientational momentum balance law*, is assumed. In the Ericksen-Leslie theory of nematic liquid crystals described in (Sonnet and Virga 2012), a similar equation is deduced with a variational procedure, starting from an assumed expression of the energy.

Of interest is the special case in which the orientation of the directors coincides with the orientation of the material elements

$$\nu^\alpha = \nabla v d^\alpha, \quad \nu_i^\alpha = v_{i,j} d_j^\alpha. \quad (71)$$

That is, the positions of the atoms in the crystal lattice follow the macroscopic deformation. This is known as the *Cauchy-Born hypothesis*. In this case, the generalized deformations ψ^α and Ψ^α take the form

$$\psi^\alpha = \nabla^S v d^\alpha, \quad \Psi^\alpha = \nabla \nabla v d^\alpha + \nabla^S v \nabla d^\alpha, \quad (72)$$

and the power (69) reduces to

$$P_{int}(\Pi, v) = \int_{\Pi} \left((T + \zeta^\alpha \otimes d^\alpha + \Sigma^\alpha \nabla^T d^\alpha) \cdot \nabla^S v + (\Sigma^\alpha \otimes d^\alpha) \cdot \nabla \nabla v \right) dV. \quad (73)$$

The presence of the term $\nabla \nabla v$ characterizes the micropolar continua which obey the Cauchy-Born hypothesis as *second-gradient continua*. These continua will be discussed in Subsection 2.5.

Also of interest are the *Cosserat continua*. They are continua in which the vectors d^α form an orthonormal triple, which preserves orthonormality during the deformation. The virtual variations ν^α are

$$\nu^\alpha = Y d^\alpha, \quad \nu_i^\alpha = Y_{ij} d_j^\alpha, \quad (74)$$

where Y is a skew-symmetric tensorial field. In this case, the generalized deformations become

$$\begin{aligned} \psi^\alpha &= (Y - \nabla^W v) d^\alpha, & \psi_i^\alpha &= (Y - \nabla^W v)_{ik} d_k^\alpha, \\ \Psi^\alpha &= -d^\alpha \nabla Y + (Y - \nabla^W v) \nabla d^\alpha, & \Psi_{ik}^\alpha &= Y_{ij,k} d_j^\alpha + (Y - \nabla^W v)_{ij} d_{j,k}^\alpha. \end{aligned} \quad (75)$$

The internal power (69) takes the form

$$P_{int}(\Pi, v, Y) = \int_{\Pi} \left(T \cdot \nabla^S v + (\zeta^\alpha \otimes d^\alpha + \Sigma^\alpha \nabla^T d^\alpha) \cdot (Y - \nabla^W v) - (d^\alpha \otimes \Sigma^\alpha) \cdot \nabla Y \right) dV,$$

and, by the indifference condition (68), it further reduces to

$$P_{int}(\Pi, v, Y) = \int_{\Pi} \left(T^S \cdot \nabla^S v - T^W \cdot (Y - \nabla^W v) - (d^\alpha \otimes \Sigma^\alpha) \cdot \nabla Y \right) dV. \quad (76)$$

Thus, the internal forces appropriate to a Cosserat continuum are the symmetric and skew-symmetric parts T^S , T^W of T , plus the third-order tensor $d^\alpha \otimes \Sigma^\alpha$. Using the vectors associated with the skew-symmetric tensors

T^W , Y and $\nabla^W v$, the last two products in (76) can be reduced to the more familiar products of moments (*couple-stresses*) by the corresponding rotations.²⁶

2.4 Continua with tensorial microstructure

An example of a continuum with a single tensorial order parameter is the tensorial model for strain-gradient plasticity. For this model, the order parameter is the plastic strain tensor, and its variation is the plastic part $\nabla^p v$ of the decomposition (61) of the displacement gradient. In the traditional approach followed in the literature,²⁷ the external power is assumed to have the form

$$P_{ext}(\Pi, v, \nabla^p v) = \int_{\Pi} (b \cdot v + B \cdot \nabla^p v) dV + \int_{\partial\Pi} (s \cdot v + S \cdot \nabla^p v) dA,$$

with B , S , and $\nabla^p v$ second-order tensors. In the approach based on bounded Cauchy fluxes the pseudobalance equations (47) and (50) hold, with the latter rewritten in the form

$$\int_{\partial\Pi} S dA + \int_{\Pi} \Phi dV = 0.$$

From them follow the relations

$$s = Tn, \quad S = \mathbb{T}n, \quad (77)$$

with \mathbb{T} a third-order tensor, and the local pseudobalance equations

$$\operatorname{div}T + f = 0, \quad \operatorname{div}\mathbb{T} + \Phi = 0. \quad (78)$$

From them, an internal power of the form

$$P_{int}(\Pi, v, \nabla^p v) = \int_{\Pi} (T \cdot \nabla v + (B - \Phi) \cdot \nabla^p v + \mathbb{T} \cdot \nabla \nabla^p v) dV, \quad (79)$$

is deduced. The tensor $\nabla^p v$ is invariant under changes of observer.²⁸ Then the rotational indifference condition

$$P_{int}(\Pi, Wx, 0) = 0 \quad (80)$$

²⁶(Aero & Kuvshinskii 1960), (Grioli 1960), (Mindlin & Tiersten 1962), (Toupin 1964).

For a detailed deduction see (Del Piero 2013a).

²⁷(Rice 1971), (Fleck & Hutchinson 2001), (Gurtin 2003a), (Gudmundson 2004), (Fleck & Willis 2009b).

²⁸See e.g. (Gurtin et al. 2010), Sect. 91.5.

requires the symmetry of T .

Equation (79) is the most general expression of the internal power for continua with tensorial microstructure. Let us show a couple of examples in which particular forms have been obtained by mixing structural properties and constitutive assumptions. In the model for small-deformation viscoplasticity of (Gurtin 2003a), it is assumed that

$$P_{int}(\Pi, v, \nabla^p v) = \int_{\Pi} (T \cdot \nabla^e v + T^p \cdot \nabla^p v + \mathbb{T} \cdot \nabla \nabla^p v) dV, \quad (81)$$

with T^p a second-order tensor, the *microstress*, and \mathbb{T} a third-order tensor, the *polar microstress*. From it, using the equation of virtual power, the microforce balance equation

$$B + \operatorname{div} \mathbb{T} = T^p - T \quad (82)$$

is deduced. We observe that, by the decomposition (61) of ∇v and the symmetry of T ,

$$P_{int}(\Pi, v, \nabla^p v) = \int_{\Pi} (T \cdot \nabla^S v + (T^p - T) \cdot \nabla^p v + \mathbb{T} \cdot \nabla \nabla^p v) dV. \quad (83)$$

This is the power (79), with T symmetric and with

$$B - \Phi = T^p - T. \quad (84)$$

Then the microforce balance equation coincides with the pseudobalance equation (78)₂.

In (Gurtin 2003a), the supplementary assumption

$$P_{int}(\Pi, 0, W) = 0, \quad (85)$$

called *relaxational isotropy*, is made to ensure the isotropy of the relaxed configuration. This is an assumption of material symmetry, and not an indifference requirement. When applied to the power (83), this condition yields the symmetry of $(T^p - T)$ and, therefore, of T^p . For $B = 0$, the symmetry of Φ follows from (84), and the symmetry of \mathbb{T} with respect to the first two subscripts

$$\mathbb{T}_{ijk} = \mathbb{T}_{jik}$$

follows from (78)₂. After decomposing $\nabla^p v$ and $\nabla^e v$ into the sum of their symmetric and skew-symmetric parts

$$\nabla^p v = D^p + W^p, \quad \nabla^e v = D^e + W^e,$$

and observing that $\nabla^S v = D^p + D^e$, we obtain

$$\begin{aligned} P_{int}(\Pi, v, D^p) &= \int_{\Pi} (T \cdot (D^p + D^e) + (T^p - T) \cdot D^p + \mathbb{T} \cdot \nabla D^p) dV \\ &= \int_{\Pi} (T \cdot D^e + T^p \cdot D^p + \mathbb{T} \cdot \nabla D^p) dV. \end{aligned} \quad (86)$$

Thus, a consequence of assumption (85) is that the *plastic spin* W^p and the *elastic spin* W^e do not contribute to the internal power.

This conclusion is not acceptable in general. Indeed, it has been recognized that the dissipation due to the plastic spin can be responsible of appreciable size effects.²⁹ A model including the dissipation due to the plastic spin is given in (Gurtin 2004). In this model assumption (85) is removed, and for the last product in (81) it is assumed that

$$\mathbb{T} \cdot \nabla \nabla^p v = T^q \cdot \text{curl} \nabla^p v, \quad \mathbb{T}_{ijk} \nabla^p v_{ij,k} = T_{hi}^q e_{hjk} \nabla^p v_{ij,k}, \quad (87)$$

where T^q is a second-order tensor and $\text{curl} \nabla^p v$ is a virtual variation of the Burgers tensor. This is the same as to assume that

$$\mathbb{T}_{ijk} = e_{hjk} T_{hi}^q, \quad T_{hi}^q = \frac{1}{2} e_{hjk} \mathbb{T}_{ijk}, \quad (88)$$

that is, that \mathbb{T} is skew-symmetric with respect to the last two subscripts, and that T^q is the second-order tensor associated with \mathbb{T} . With this assumption, using again the equality $\nabla^S v = D^p + D^e$ and observing that

$$T \cdot \nabla^p v = T \cdot D^p,$$

by the symmetry of T , the internal power (83) reduces to

$$\begin{aligned} P_{int}(\Pi, v, \nabla^p v) &= \int_{\Pi} (T \cdot \nabla^S v + (T^p - T) \cdot \nabla^p v + T^q \cdot \text{curl} \nabla^p v) dV \\ &= \int_{\Pi} (T \cdot D^e + T^p \cdot \nabla^p v + T^q \cdot \text{curl} \nabla^p v) dV. \end{aligned} \quad (89)$$

Moreover, by the identity

$$\text{div} \mathbb{T} = -(\text{curl} T^{qT})^T, \quad \mathbb{T}_{ijk,k} = e_{hjk} T_{hi,k}^q = -(\text{curl} T^{qT})_{ji},$$

which follows from (88), the microforce balance equation (82) takes the form

$$B - \text{curl} (T^{qT})^T = T^p - T. \quad (90)$$

²⁹(Niordson & Hutchinson 2003), (Fleck & Willis 2009b), (Bardella 2010).

2.5 Second-gradient continua

A second-gradient continuum is a particular continuum with tensorial microstructure, whose unique order parameter coincides with the gradient of the macroscopic deformation u . Its virtual velocity is

$$\nu = \nabla v, \quad (91)$$

and the external power is

$$P_{ext}(\Pi, v) = \int_{\Pi} (b \cdot v + B \cdot \nabla v) dV + \int_{\partial\Pi} (s \cdot v + S \cdot \nabla v) dA. \quad (92)$$

This is a special case of the power (49), with the body microforce and the surface microtraction given by the second-order tensors B and S , respectively. From the representations (77) of s and S , the pseudobalance equations (78) and the internal power

$$P_{int}(\Pi, v) = \int_{\Pi} ((T + B - \Phi) \cdot \nabla^S v + \mathbb{T} \cdot \nabla \nabla v) dV \quad (93)$$

follow. Note that ∇v has been replaced by $\nabla^S v$, because the condition of rotational indifference (80) here requires the symmetry of $(T + B - \Phi)$. Then T is not symmetric in general, its skew-symmetric part is

$$T^W = \Phi^W - B^W,$$

and the local balance (7) of the linear momentum reduces to³⁰

$$\operatorname{div} T^S + \operatorname{div} (\Phi^W - B^W) + b = 0. \quad (94)$$

For a continuum with vectorial microstructure obeying the Cauchy-Born hypothesis (71), the external power has the form

$$\int_{\Pi} (b \cdot v + (\beta^\alpha \otimes d^\alpha) \cdot \nabla v) dV + \int_{\partial\Pi} (s \cdot v + (\sigma^\alpha \otimes d^\alpha) \cdot \nabla v) dA, \quad (95)$$

with sum over the superscripts α . This is the power of a particular second-gradient continuum, with

$$B = \beta^\alpha \otimes \nu^\alpha, \quad S = \sigma^\alpha \otimes \nu^\alpha.$$

The presence in (93) of a second-gradient term causes some problems in the formulation of the boundary conditions for the problem of motion. Indeed,

³⁰(Germain 1973a), Eqs. (35) and (36).

the displacement gradient at the boundary has a normal and a tangential component, and the tangential component is determined by the values of v at the boundary. By consequence, boundary conditions of place can be prescribed only to v and to the normal component of ∇v .

For a three-dimensional body Ω , to eliminate the tangential component at the boundary take a local orthonormal reference frame (e^α, e^n) , where e^α , $\alpha \in \{1, 2\}$, are tangent vectors, and e^n is the exterior normal n to $\partial\Omega$. Consider the decomposition

$$S \cdot \nabla v = S_{ij}v_{i,j} = S_{in}v_{i,n} + S_{i\alpha}v_{i,\alpha} = Sn \cdot \nabla_n v + Se^\alpha \cdot \nabla_\alpha v. \quad (96)$$

of the product $S \cdot \nabla v$ into a normal and a tangential part. By the Gauss-Green formula,

$$\int_{\partial\Omega} Se^\alpha \cdot \nabla_\alpha v \, dA = \int_{\partial\Omega} S_{i\alpha}v_{i,\alpha} \, dA = - \int_{\partial\Omega} S_{i\alpha,\alpha}v_i \, dA = - \int_{\partial\Omega} \operatorname{div}_\alpha S \cdot v \, dA, \quad (97)$$

and, therefore,

$$\int_{\partial\Omega} S \cdot \nabla v \, dA = \int_{\partial\Omega} (Sn \cdot \nabla_n v - \operatorname{div}_\alpha S \cdot v) \, dA. \quad (98)$$

The derivatives appearing in the operator $(\operatorname{div}_\alpha)$ are distributional derivatives. That is, they may contain concentrated terms of the Dirac type. This occurs at the *edge lines*, which are singular lines of the surface $\partial\Omega$ at which the exterior normal and consequently the tangent plane, are discontinuous. Indeed, forces per unit length, called *edge forces*, appear on these lines.³¹ But these forces are only apparent, because they are due to the representation of the power in a discontinuous local basis, and not to *real* forces applied from the exterior.³² Indeed, equation (98) shows that edge forces are present even for very regular surface microtractions S .

Some authors believe that the presence of edge forces requires a reformulation of the theorems of Noll and Cauchy proved in Subsection 1.2. This seems not to be necessary, as long as the regularity assumed for the Cauchy fluxes excludes singularities of the surface microtraction S . Indeed,

³¹Edge forces were considered in (Toupin 1962), (Mindlin 1964), (Germain 1973b). For more recent developments see (Noll & Virga 1990), (Dell'Isola & Seppecher 1995), (Degiovanni et al. 2006), (Podio-Guidugli & Vianello 2010).

³²As stated in (Noll & Virga 1990), "edge interactions should not be confused with external actions concentrated along curves".

in this case, the tensorial versions of Noll's and Cauchy's theorems leading to the relations (31) hold.³³

On the contrary, both more regular regions and generalized versions of Noll's and Cauchy's theorems are required when the external actions include forces distributed on lines or concentrated at isolated points.³⁴ But, in spite of the large literature on the subject, it seems that a complete theory of higher-order continua in the presence of singular external forces has not yet been formulated.

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³³It may happen that very regular surface tractions at the boundary produce stress concentrations at interior points, lines, or surfaces, see e.g. (Podio Guidugli 2004) and (Lucchesi et al. 2005, 2009). At my knowledge, general a priori conditions excluding such concentrations are not known.

³⁴For Noll's theorem see (Dell'Isola et al. 2012). For proofs of the Cauchy theorem without the tetrahedron argument see (Šilhavý 1985, 1991, 2008), (Fosdick & Virga 1989), (Marzocchi & Musesti 2003).

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