The micromorphic approach to plasticity and phase transformation

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Objectives

The objectives of this presentation are two-fold:

- propose a systematic procedure to extend standard elastoviscoplasticity models to include:
  - ⋆ size effects in the hardening behaviour of materials (grain size effects...)
  - ⋆ regularization properties in the softening behaviour (strain localization...)
- unify the "zoology" of generalized continuum models:
  - ⋆ "Classical" generalized continua: Cosserat, second gradient, micromorphic media
    (Mindlin, 1964; Eringen and Suhubi, 1964; Mindlin and Eshel, 1968)
  - ⋆ strain gradient plasticity, "implicit gradient approach"...
    (Aifantis, 1987; Fleck and Hutchinson, 2001; Gurtin, 2003; Engelen et al., 2003)
- establish links between generalized continuum mechanics and phase field approaches

Plan

1. The micromorphic approach to plasticity
   - Continuum thermomechanics
   - Full micromorphic and microstrain theories
2. Microstrain gradient plasticity
   - Gradient of plastic microstrain
   - Consistency condition
   - Anisothermal strain gradient plasticity
3. Internal constraint in the micromorphic approach
4. Microdiffusion and phase field approach

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State space

- **observable and controllable variables** (temperature, strain...)

$$\{T, \quad \underset{\sim}{\varepsilon}\}$$

- **internal degrees of freedom** (controllable variables that account for some aspects of the microstructre)

$$\{\alpha, \quad \nabla\alpha\}$$

  they have associated stresses and α or its associated force can be prescribed at the boundary

- **internal variables** are the remembrance of internal degrees of freedom; they cannot be controlled

$$\{\alpha\}$$

The micromorphic approach (1)

- Start from an initial classical elastoviscoplastic model with internal variables

$$DOF0 = \{\underline{\mathbf{u}}\}, \quad STATE0 = \{\underset{\sim}{\mathbf{F}}, \quad T, \quad \alpha\}$$

- Select one variable $\phi \in STATE0$ and introduce the associated micromorphic variable $^{\chi}\phi$ as an additional degree of freedom and, possibly, state variable:

$$DOF = \{\underline{\mathbf{u}}, \quad {}^{\chi}\phi\}, \quad STATE = \{\underset{\sim}{\mathbf{F}}, \quad T, \quad \alpha, \quad {}^{\chi}\phi, \quad \boldsymbol{\nabla}\, {}^{\chi}\phi\}$$

- Extend the power of internal forces

$$\mathcal{P}^{(i)}(\underline{\mathbf{v}}^{\star}, {}^{\chi}\dot{\phi}^{\star}) = -\int_{\mathcal{D}} p^{(i)}(\underline{\mathbf{v}}^{\star}, {}^{\chi}\dot{\phi}^{\star})\, dV$$

$$p^{(i)}(\underline{\mathbf{v}}^{\star}, {}^{\chi}\dot{\phi}^{\star}) = \underset{\sim}{\boldsymbol{\sigma}} : \boldsymbol{\nabla}\underline{\mathbf{v}}^{\star} + a\, {}^{\chi}\dot{\phi}^{\star} + \underline{\mathbf{b}} \,.\, \boldsymbol{\nabla}\, {}^{\chi}\dot{\phi}^{\star}$$

$a, \underline{\mathbf{b}}$ generalized stresses, *microforces* (Gurtin, 1996)

- Derive additional balance equation and boundary conditions

$$\mathrm{div}\, \underline{\mathbf{b}} - a = 0, \quad \forall \underline{\mathbf{x}} \in \Omega, \quad \underline{\mathbf{b}} \,.\, \underline{\mathbf{n}} = a^c, \forall \underline{\mathbf{x}} \in \partial\Omega$$

The micromorphic approach (2)

- More generally, in the presence of volume generalized forces:

$$\mathrm{div}\,(\underline{\mathbf{b}} - \underline{\mathbf{b}}^{\,e}) - a + a^e = 0, \quad \forall \underline{\mathbf{x}} \in \Omega, \quad (\underline{\mathbf{b}} - \underline{\mathbf{b}}^{\,e}).\underline{\mathbf{n}} = a^c, \forall \underline{\mathbf{x}} \in \partial\Omega$$

- Enhance the local balance of energy and the entropy inequality

$$\rho\dot{\epsilon} = p^{(i)} - \mathrm{div}\,\underline{\mathbf{q}} + \rho r, \quad -\rho(\dot{\psi} + \eta\dot{T}) + p^{(i)} - \frac{\underline{\mathbf{q}}}{T}.\boldsymbol{\nabla} T \geq 0$$

- Consider the constitutive functionals:

$$\begin{aligned}
\psi &= \hat{\psi}(\underset{\sim}{\mathbf{F}}^e, T, \alpha, {}^\chi\phi, \boldsymbol{\nabla}^\chi\phi), \;\; \eta = \hat{\eta}(\underset{\sim}{\mathbf{F}}^e, T, \alpha, {}^\chi\phi, \boldsymbol{\nabla}^\chi\phi) \\
\underset{\sim}{\boldsymbol{\sigma}} &= \hat{\underset{\sim}{\boldsymbol{\sigma}}}(\underset{\sim}{\mathbf{F}}^e, T, \alpha, {}^\chi\phi, \boldsymbol{\nabla}^\chi\phi) \\
a &= \hat{a}(\underset{\sim}{\mathbf{F}}^e, T, \alpha, {}^\chi\phi, \boldsymbol{\nabla}^\chi\phi), \quad \underline{\mathbf{b}} = \hat{\underline{\mathbf{b}}}(\underset{\sim}{\mathbf{F}}^e, T, \alpha, {}^\chi\phi, \boldsymbol{\nabla}^\chi\phi)
\end{aligned}$$

- Derive the state laws (Coleman and Noll, 1963)

$$\underset{\sim}{\boldsymbol{\sigma}} = \rho\frac{\partial\hat{\psi}}{\partial\underset{\sim}{\mathbf{F}}^e}.\underset{\sim}{\mathbf{F}}^{eT}, \; \eta = -\frac{\partial\hat{\psi}}{\partial T}, \; X = \rho\frac{\partial\hat{\psi}}{\partial\alpha}, \quad a = \frac{\partial\hat{\psi}}{\partial\,{}^\chi\phi}, \quad \underline{\mathbf{b}} = \frac{\partial\hat{\psi}}{\partial\boldsymbol{\nabla}\,{}^\chi\phi}$$

- Residual dissipation $\quad D^{res} = W^p - X\dot{\alpha} - \dfrac{\underline{\mathbf{q}}}{T}.\boldsymbol{\nabla} T \geq 0$

The micromorphic approach (3)

- Take a simple quadratic potential

$$\psi(\underset{\sim}{\mathbf{F}}, T, \alpha, {}^{\chi}\phi, \nabla\, {}^{\chi}\phi) = \psi_1(\underset{\sim}{\mathbf{F}}, \alpha, T) + \psi_2(e = \phi - {}^{\chi}\phi, \nabla\, {}^{\chi}\phi, T)$$

$$\rho\psi_2 = \frac{1}{2} H_\chi (\phi - {}^{\chi}\phi)^2 + \frac{1}{2} A \nabla\, {}^{\chi}\phi . \nabla\, {}^{\chi}\phi$$

$$a = \rho \frac{\partial \psi}{\partial\, {}^{\chi}\phi} = -H_\chi(\phi - {}^{\chi}\phi), \quad \underline{\mathbf{b}} = \rho \frac{\partial \psi}{\partial \nabla\, {}^{\chi}\phi} = A \nabla\, {}^{\chi}\phi$$

- Simple form of the partial differential equation (homogeneous, isothermal...)

$$a = \mathrm{div}\, \underline{\mathbf{b}} \implies {}^{\chi}\phi - \frac{A}{H_\chi} \Delta\, {}^{\chi}\phi = \phi$$

Helmholtz equation with a minus sign and a source term

- Coupling modulus H_χ and characteristic length of the medium

$$l_c^2 = \frac{A}{H_\chi}$$

Stability $\quad H_\chi > 0, \quad A > 0$

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Micromorphic continuum

Micromorphic continuum according to (Eringen and Suhubi, 1964; Mindlin, 1964)

- Select variable:

$$\phi \equiv \underset{\sim}{\mathbf{F}}, \quad {}^{\chi}\phi \equiv \underset{\sim}{\chi}$$

$$p^{(i)} = \underset{\sim}{\boldsymbol{\sigma}} : \boldsymbol{\nabla}\underline{\dot{\mathbf{u}}} + \underset{\sim}{\mathbf{a}} : \underset{\sim}{\dot{\chi}} + \underset{\approx}{\mathbf{B}} \vdots \boldsymbol{\nabla}\underset{\sim}{\dot{\chi}}$$

- application of the principle of (infinitesimal) material frame indifference, (infinitesimal) change of observer of rate $\underset{\sim}{\mathbf{w}}$:

$$\boldsymbol{\nabla}\underline{\dot{\mathbf{u}}} \Longrightarrow \boldsymbol{\nabla}\underline{\dot{\mathbf{u}}} + \underset{\sim}{\mathbf{w}}, \quad \underset{\sim}{\chi} \Longrightarrow \underset{\sim}{\chi} + \underset{\sim}{\mathbf{w}}$$

$\Longrightarrow \underset{\sim}{\boldsymbol{\sigma}} + \underset{\sim}{\mathbf{a}}$ must be symmetric. Rewrite the virtual power:

$$p^{(i)} = \underset{\sim}{\boldsymbol{\sigma}} : \underset{\sim}{\dot{\varepsilon}} + \underset{\sim}{\mathbf{s}} : (\boldsymbol{\nabla}\underline{\dot{\mathbf{u}}} - \underset{\sim}{\dot{\chi}}) + \underset{\approx}{\mathbf{S}} \vdots \boldsymbol{\nabla}\underset{\sim}{\dot{\chi}}$$

- two balance equations:

$$\mathrm{div}\,(\underset{\sim}{\boldsymbol{\sigma}} + \underset{\sim}{\mathbf{s}}) + \rho\underline{\mathbf{f}} = 0, \quad \mathrm{div}\,\underset{\approx}{\mathbf{S}} + s = 0$$

Microstrain continuum

Microstrain continuum after (Forest and Sievert, 2006)

- Select

$$\phi \equiv \underset{\sim}{\mathbf{C}} = \underset{\sim}{\mathbf{F}}^T . \underset{\sim}{\mathbf{F}}, \quad {}^\chi\phi \equiv {}^\chi\underset{\sim}{\mathbf{C}}, \quad \text{or} \quad \phi \equiv \underset{\sim}{\varepsilon}, \quad {}^\chi\phi \equiv {}^\chi\underset{\sim}{\varepsilon}$$

$$p^{(i)} = \underset{\sim}{\sigma} : \dot{\underset{\sim}{\varepsilon}} + \underset{\sim}{\mathbf{a}} : {}^\chi\dot{\underset{\sim}{\varepsilon}} + \underset{\approx}{\mathbf{b}} \vdots \nabla {}^\chi\underset{\sim}{\varepsilon}$$

- Constitutive coupling between macro and microstrain via the relative strain

$$\underset{\sim}{\mathbf{e}} := \underset{\sim}{\varepsilon} - {}^\chi\underset{\sim}{\varepsilon}$$

$$\psi(\underset{\sim}{\varepsilon}^e, \quad T, \quad \alpha, \quad \underset{\sim}{\mathbf{e}} := \underset{\sim}{\varepsilon} - {}^\chi\underset{\sim}{\varepsilon}, \quad \underset{\approx}{\mathbf{K}} := \nabla {}^\chi\underset{\sim}{\varepsilon})$$

- Take a quadratic potential

$$\underset{\sim}{\mathbf{a}} = H_\chi \underset{\sim}{\mathbf{e}}, \quad \underset{\approx}{\mathbf{b}} = A \nabla {}^\chi\underset{\sim}{\varepsilon}$$

- Extra–balance equation

$${}^\chi\underset{\sim}{\varepsilon} - l_c^2 \Delta {}^\chi\underset{\sim}{\varepsilon} = \underset{\sim}{\varepsilon}, \quad \text{with} \quad l_c^2 = \frac{A}{H_\chi}$$

example: microfoams (Dillard et al., 2006)

Cosserat continuum

Cosserat continuum

$$\phi = \underset{\sim}{\mathbf{R}}, \quad {}^{\chi}\phi \equiv {}^{\chi}\underset{\sim}{\mathbf{R}}$$

$$p^{(i)} = \underset{\sim}{\boldsymbol{\sigma}}^{s} : \dot{\underset{\sim}{\boldsymbol{\varepsilon}}} - \underset{\sim}{\boldsymbol{\sigma}}^{a} : ((\boldsymbol{\nabla}\underline{\dot{\mathbf{u}}})^{a} - \dot{\underset{\sim}{\mathbf{R}}}.\underset{\sim}{\mathbf{R}}^{T}) + \underset{\sim}{\mathbf{M}} : \dot{\underset{\sim}{\boldsymbol{\kappa}}}$$

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General scalar microstrain gradient plasticity

- Classical and generalized plasticity

$$DOF0 = \{\underline{\mathbf{u}}\} \quad STATE0 = \{\underset{\sim}{\varepsilon}^e, \quad p, \quad \alpha\}$$

$$\phi \equiv p, \quad {}^\chi\phi \equiv {}^\chi p$$

$$DOF = \{\underline{\mathbf{u}}, \quad {}^\chi p\} \quad STATE = \{\underset{\sim}{\varepsilon}^e, \quad p, \quad \alpha, \quad {}^\chi p, \quad \boldsymbol{\nabla}^\chi p\}$$

- Extra balance equation

$$p^{(i)} = \underset{\sim}{\boldsymbol{\sigma}} : \dot{\underset{\sim}{\varepsilon}} + a\,{}^\chi\dot{p} + \underline{\mathbf{b}}\,.\boldsymbol{\nabla}\,{}^\chi\dot{p}, \quad p^{(c)} = \underline{\mathbf{t}}\,.\dot{\underline{\mathbf{u}}} + a^c\,{}^\chi\dot{p}$$

$$\mathrm{div}\,\underline{\mathbf{b}} - a = 0, \quad \forall \underline{\mathbf{x}} \in \Omega, \quad \underline{\mathbf{b}}\,.\underline{\mathbf{n}} = a^c, \quad \forall \underline{\mathbf{x}} \in \partial\Omega$$

- State laws

$$\underset{\sim}{\varepsilon} = \underset{\sim}{\varepsilon}^e + \underset{\sim}{\varepsilon}^p$$

$$\underset{\sim}{\boldsymbol{\sigma}} = \rho\frac{\partial\psi}{\partial\underset{\sim}{\varepsilon}^e}, \quad R = \rho\frac{\partial\psi}{\partial p}, \quad X = \rho\frac{\partial\psi}{\partial\alpha}, \quad a = \rho\frac{\partial\psi}{\partial\,{}^\chi p}, \quad \underline{\mathbf{b}} = \rho\frac{\partial\psi}{\partial\boldsymbol{\nabla}\,{}^\chi p}$$

- Evolution laws

$$D^{res} = \underset{\sim}{\boldsymbol{\sigma}} : \dot{\underset{\sim}{\varepsilon}}^p - R\dot{p} - X\dot{\alpha} \geq 0$$

$$\dot{\underset{\sim}{\varepsilon}}^p = \dot{\lambda}\frac{\partial f}{\partial\underset{\sim}{\boldsymbol{\sigma}}}, \quad \dot{p} = -\dot{\lambda}\frac{\partial f}{\partial R}, \quad \dot{\alpha} = -\dot{\lambda}\frac{\partial f}{\partial X}$$

Simplified scalar microstrain gradient plasticity

- Quadratic free energy potential

$$\rho\psi(\underset{\sim}{\varepsilon}^e, p, {}^\chi p, \nabla {}^\chi p) = \frac{1}{2}\underset{\sim}{\varepsilon}^e : \underset{\approx}{\mathbf{\Lambda}} : \underset{\sim}{\varepsilon}^e + \frac{1}{2}Hp^2 + \frac{1}{2}H_\chi(p - {}^\chi p)^2 + \frac{1}{2}\nabla {}^\chi p . \underset{\sim}{\mathbf{A}} . \nabla {}^\chi p$$

- Constitutive equations

$$\underset{\sim}{\sigma} = \underset{\approx}{\mathbf{\Lambda}} : \underset{\sim}{\varepsilon}^e, \; a = -H_\chi(p - {}^\chi p), \; \underline{\mathbf{b}} = \underset{\sim}{\mathbf{A}} . \nabla {}^\chi p, \; R = (H + H_\chi)p - H_\chi {}^\chi p$$

- Substitution of constitutive equation into extra balance equation

$${}^\chi p - \frac{1}{H_\chi}\mathrm{div}\,(\underset{\sim}{\mathbf{A}} . \nabla {}^\chi p) = p$$

- Homogeneous and isotropic materials $\underset{\sim}{\mathbf{A}} = A\underset{\sim}{\mathbf{1}}$

$${}^\chi p - \frac{A}{H_\chi}\Delta {}^\chi p = p, \quad \text{b.c.} \quad \nabla {}^\chi p . \underline{\mathbf{n}} = a^c$$

same partial differential equation as in the *implicit gradient–enhanced elastoplasticity* with $a^c = 0$ (Engelen et al., 2003)

Link to Aifantis strain gradient plasticity

- Yield function

$$f(\underset{\sim}{\sigma}, R) = \sigma_{eq} - \sigma_Y - R$$

- Hardening law

$$R = \frac{\partial \psi}{\partial p} = (H + H_\chi) p - H_\chi {}^\chi p$$

- Under plastic loading

$$\sigma_{eq} = \sigma_Y + H\, {}^\chi p - A(1 + \frac{H}{H_\chi}) \Delta\, {}^\chi p$$

compare with Aifantis model (Aifantis, 1987)

$$\sigma_{eq} = \sigma_Y + R(p) - c^2 \Delta p$$

The equivalence is obtained for $H_\chi = \infty$ (internal constraint):

$${}^\chi p \simeq p, \quad A = c^2$$

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Consistency condition

- Consistency condition

$$\dot{f} = \frac{\partial f}{\partial \underset{\sim}{\boldsymbol{\sigma}}} : \dot{\underset{\sim}{\boldsymbol{\sigma}}} + + \frac{\partial f}{\partial R}\dot{R}$$
$$= \frac{\partial \sigma_{eq}}{\partial \sigma} : \underset{\approx}{\boldsymbol{\Lambda}} : (\dot{\underset{\sim}{\varepsilon}} - \dot{\underset{\sim}{\varepsilon}}^{p}) - \frac{\partial R}{\partial p}\dot{p} - \frac{\partial R}{\partial\, {}^{\chi}p}\, {}^{\chi}\dot{p} = 0$$

- Plastic multiplier

$$\dot{p} = \frac{\underset{\sim}{\mathbf{N}} : \underset{\approx}{\boldsymbol{\Lambda}} : \dot{\underset{\sim}{\varepsilon}} - \dfrac{\partial R}{\partial\, {}^{\chi}p}\, {}^{\chi}\dot{p}}{\underset{\sim}{\mathbf{N}} : \underset{\approx}{\boldsymbol{\Lambda}} : \underset{\sim}{\mathbf{N}} + \dfrac{\partial R}{\partial p}}, \quad \text{with} \quad \underset{\sim}{\mathbf{N}} = \frac{\partial \sigma_{eq}}{\partial \underset{\sim}{\boldsymbol{\sigma}}}$$

where $\dot{\underset{\sim}{\varepsilon}}$ and ${}^{\chi}\dot{p}$ are controllable variable.

- Even though the yield condition can be written as a partial differential equation, there is no need for a variational formulation of the consistency condition contrary to (Mühlhaus and Aifantis, 1991; Liebe et al., 2001). There is no need for a plastic front tracking technique. The plastic microstrain ${}^{\chi}p$ and the generalized traction $\underline{\mathbf{b}}\,.\underline{\mathbf{n}}$ are continuous across the elastic/plastic domain.

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Thermal effects

- For temperature dependent parameters

$$a = \operatorname{div} \underline{\mathbf{b}} = \operatorname{div}(A \nabla {}^{\chi}p) = A \Delta {}^{\chi}p + \frac{\partial A}{\partial T} \nabla T . \nabla^{\chi} p$$

$${}^{\chi}p - \frac{A}{H_{\chi}} \Delta {}^{\chi}p - \frac{1}{H_{\chi}} \frac{\partial A}{\partial T} \nabla T . \nabla {}^{\chi}p = p$$

- Consistency condition

$$\dot{p} = \frac{\underset{\sim}{\mathbf{N}} : \underset{\approx}{\mathbf{\Lambda}} : (\dot{\underset{\sim}{\varepsilon}} - \dot{\underset{\sim}{\varepsilon}}^{th}) - \dfrac{\partial R}{\partial {}^{\chi}p} {}^{\chi}\dot{p} - \dfrac{\partial R}{\partial T} \dot{T}}{\underset{\sim}{\mathbf{N}} : \underset{\approx}{\mathbf{\Lambda}} : \underset{\sim}{\mathbf{N}} + \dfrac{\partial R}{\partial p}}$$

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Internal constraint and gradient of internal variable approach

- Impose the internal constraint that

$$^{\chi}\phi \simeq \phi \quad \Longrightarrow \quad \underline{\mathbf{K}} \simeq \nabla \phi$$

Then, the generalized stress a becomes a Lagrange multiplier.

- Examples
  - ⋆ $\phi \equiv \underset{\sim}{\mathbf{F}}$ second gradient model (Mindlin, 1965)
  - ⋆ $\phi \equiv p$ Aifantis model (Aifantis, 1987; Fleck and Hutchinson, 2001)
  - ⋆ $\phi \equiv \underset{\sim}{\varepsilon}^p$ strain gradient plasticity (Forest and Sievert, 2003; Gurtin, 2003)

$$p^{(i)} = \underset{\sim}{\boldsymbol{\sigma}} : \dot{\underset{\sim}{\varepsilon}}^e + \underset{\sim}{\mathbf{s}} : \dot{\underset{\sim}{\varepsilon}}^p + \underset{\approx}{\mathbf{S}} \,\vdots\, \dot{\underset{\approx}{\varepsilon}}^p, \quad \operatorname{div} \underset{\approx}{\mathbf{S}} = \underset{\sim}{\mathbf{s}} - \underset{\sim}{\boldsymbol{\sigma}}^{dev}$$

The yield condition becomes a PDE (Aifantis, Fleck–Hutchinson, Gurtin):

$$\sigma_{eq} = \sigma_Y + Hp - A\Delta p$$

What does the yield criterion become?

- What do the boundary conditions become?
  - ⋆ For the second gradient theory, intricate b.c. involving surface curvature
  - ⋆ For gradient of plastic strain, $\underset{\approx}{\mathbf{S}}.\underline{\mathbf{n}} = \underset{\sim}{\mathbf{m}}$

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Microdiffusion (1)

Putting Geers' approach of viscoplasticity and Cahn–Hilliard diffusion within the micromorphic framework (Ubachs et al., 2004)

- Mass concentration and microconcentration

$$\phi \equiv c, \quad {}^{\chi}\phi \equiv {}^{\chi}c, \quad STATE = \{c, {}^{\chi}c, \quad \nabla\, {}^{\chi}c\}$$

- Additional power due to microdiffusion (compare: there is no power produced by classical diffusion!)

$$p^{(i)} = a\, {}^{\chi}\dot{c} + \underline{\mathbf{b}} \,.\nabla\, {}^{\chi}\dot{c}, \quad a = \operatorname{div} \underline{\mathbf{b}}\,, \quad \underline{\mathbf{b}} \,.\underline{\mathbf{n}} \, = a^c$$

in addition to the balance of mass:

$$\rho\dot{c} = -\operatorname{div} \underline{\mathbf{J}}$$

- First and second principles (isothermal for brevity)

$$\rho\dot{\epsilon} = p^{(i)}, \quad \int_V \rho\dot{\eta}\, dV \geq \int_{\mathcal{V}} \frac{\mu\underline{\mathbf{J}}}{T}\, dS$$

mass flux $\underline{\mathbf{J}}$ and chemical potential μ

$$\rho\, T\, \dot{\eta} - \operatorname{div}(\mu\underline{\mathbf{J}}\,) \geq 0; \quad -\rho\dot{\psi} + p^{(i)} - \operatorname{div}(\mu\underline{\mathbf{J}}\,) \geq 0$$

Microdiffusion (2)

- State laws $\rho\psi(c, \quad {}^{\chi}c, \quad \nabla\,{}^{\chi}c)$

$$(a-\rho\frac{\partial\psi}{\partial\,{}^{\chi}c})\,{}^{\chi}\dot{c}+(\underline{\mathbf{b}}-\rho\frac{\partial\psi}{\partial\nabla\,{}^{\chi}c}).\nabla\,{}^{\chi}\dot{c}+\rho(\mu-\frac{\partial\psi}{\partial c})\dot{c}-\underline{\mathbf{J}}.\nabla\mu\geq 0$$

$$a=\rho\frac{\partial\psi}{\partial\,{}^{\chi}c}, \qquad \underline{\mathbf{b}}=\rho\frac{\partial\psi}{\partial\nabla\,{}^{\chi}c}, \quad \mu=\frac{\partial\psi}{\partial c}$$

- Quadratic potential

$$\rho\psi=\rho\psi_0(c)+\frac{1}{2}H_\chi(c-\,{}^{\chi}c)^2+\frac{1}{2}\alpha\nabla\,{}^{\chi}c.\nabla\,{}^{\chi}c$$

$$a=-H_\chi(c-\,{}^{\chi}c)=\mathrm{div}\,\underline{\mathbf{b}}=\alpha\Delta\,{}^{\chi}c$$

$${}^{\chi}c-\lambda^2\Delta\,{}^{\chi}c=c, \quad \lambda^2=\frac{\alpha}{H_\chi}$$

$$\mu=\rho\frac{\partial\psi_0}{\partial c}+H_\chi(c-\,{}^{\chi}c)$$

Relation to Cahn–Hilliard theory

- Mass concentration $\phi \equiv c, \quad STATE = \{c, \quad \nabla c\}$

$$p^{(i)} = a\dot{c} + \underline{\mathbf{b}}\,.\nabla\dot{c}, \quad a = \mathrm{div}\,\underline{\mathbf{b}}\,, \quad \underline{\mathbf{b}}\,.\underline{\mathbf{n}} = a^c$$

  in addition to the balance of mass $\rho\dot{c} = -\mathrm{div}\,\underline{\mathbf{J}}$ (Gurtin, 1996)

- First and second principles (isothermal for brevity) $\rho\psi(c, \quad \nabla c)$

$$\rho\dot{\epsilon} = p^{(i)}, \quad -\rho\dot{\psi} + p^{(i)} - \mathrm{div}\,(\mu\underline{\mathbf{J}}\,) \geq 0$$

$$(a + \mu - \rho\frac{\partial\psi}{\partial c})\dot{c} + (\underline{\mathbf{b}} - \rho\frac{\partial\psi}{\partial\nabla c}).\nabla\dot{c} - \underline{\mathbf{J}}\,.\nabla\mu \geq 0$$

$$\mu = \rho\frac{\partial\psi}{\partial c} - a, \qquad \underline{\mathbf{b}} = \rho\frac{\partial\psi}{\partial\nabla c}$$

- Fick's law $\underline{\mathbf{J}} = -\kappa\nabla\mu$
- Quadratic potential $\rho\psi = \rho\psi_0(c) + \frac{1}{2}\alpha\nabla c.\nabla c$ (Cahn and Hilliard, 1958)

$$\mu = \rho\frac{\partial\psi}{\partial c} - a = \rho\frac{\partial\psi}{\partial c} - \mathrm{div}\,\underline{\mathbf{b}} = \rho\frac{\partial\psi}{\partial c} - \alpha\Delta c$$

$$\rho\dot{c} = -\mathrm{div}\,\underline{\mathbf{J}} = \kappa\Delta\mu = \kappa\Delta(\rho\frac{\partial\psi}{\partial c} - \alpha\Delta c)$$

- Equivalence obtained for ${}^{\chi}c \simeq c$

$$\rho\dot{c} = \mathrm{div}\,\nabla\mu = \kappa\Delta(\rho\frac{\partial\psi_0}{\partial c} + H_\chi(c - {}^{\chi}c)) = \kappa\Delta(\rho\frac{\partial\psi_0}{\partial c} - \alpha\Delta\,{}^{\chi}c)$$

Phase field approach (1)

The phase field model as presented by (Gurtin, 1996) falls in the micromorphic approach. There are however two differences compared to the previous examples: $\phi \notin STATE0$, there is a dissipative part associated with $\dot{\phi}$

- Order parameter ϕ as additional degree of freedom in addition to mass concentration; Gurtin assumes that there is a power expenditure by variation of order parameter and its gradient (in contrast to diffusion!)

$$STATE = \{c, \quad \phi, \quad \nabla \phi\}, \quad p^{(i)} = a\dot{\phi} + \underline{\mathbf{b}} \, . \nabla \dot{\phi}$$

- Balance of mass, generalized momentum (no volume forces) and energy

$$\rho \dot{c} = -\mathrm{div}\, \underline{\mathbf{J}}\,, \quad \mathrm{div}\, \underline{\mathbf{b}} - a = 0, \quad \rho \dot{\epsilon} = p^{(i)}$$

- Exploitation of second principle *à la* Coleman–Noll

$$p^{(i)} - \rho \dot{\psi} - \mathrm{div}\, \mu \underline{\mathbf{J}} \geq 0$$

Phase field approach (2)

- Exploitation of the second principle (continued)

$$\rho(\mu - \frac{\partial \psi}{\partial c})\dot{c} + (a - \rho\frac{\partial \psi}{\partial \phi})\dot{\phi} - (\underline{\mathbf{b}} - \rho\frac{\partial \psi}{\partial \nabla \phi}).\nabla\dot{\phi} - \underline{\mathbf{J}}\,.\nabla\mu \geq 0$$

$$\rho\psi = \rho\psi_0(c,\phi) + \frac{1}{2}\alpha\nabla\phi.\nabla\phi, \quad \mu = \frac{\partial \psi}{\partial c} =, \quad \underline{\mathbf{b}} = \rho\frac{\partial \psi}{\partial \nabla \phi}$$

- Accept the dependence $a(c, \phi, \nabla\phi, \dot{\phi})$ $\quad a^{\mathrm{dis}} = a - \rho\frac{\partial \psi}{\partial \phi}$
  and choose the dissipation potential

$$\Omega(\nabla\mu, a^{\mathrm{dis}}) = \frac{1}{2}\kappa\nabla\mu.\nabla\mu + \frac{1}{2\beta}(a^{\mathrm{dis}})^2$$

$$\underline{\mathbf{J}} = -\frac{\partial \Omega}{\partial \nabla\mu} = -\kappa\nabla\mu, \quad \dot{\phi} = \frac{\partial \Omega}{\partial a^{\mathrm{dis}}} = \frac{1}{\beta}a^{\mathrm{dis}}$$

- Ginzburg–Landau (Allen–Cahn) equation

$$\beta\dot{\phi} = a^{\mathrm{dis}} = a - \rho\frac{\partial \psi_0}{\partial \phi} = \mathrm{div}\,\underline{\mathbf{b}} - \rho\frac{\partial \psi_0}{\partial \phi} = \alpha\Delta\phi - \rho\frac{\partial \psi_0}{\partial \phi}$$

  implemented in this way by Kais Ammar (2007)

Conclusions

Why the name "micromorphic approach"?

- additional degrees of freedom, generally "strain–like" variables, in the spirit of the full micromorphic continuum by Mindlin and Eringen
- coupling of macro and micro–quantities through a dependence of the free energy on a relative strain measure $e = \phi - {}^{\chi}\phi$
- additional balance equations taking the form of a Helmholtz equation with source term for a simple choice of the free energy function
- constrained micromorphic media: strain gradient plasticity and damage
- microdiffusion model that can be reduced to Cahn-Hilliard model
- applications: finite element simulations of cell–size effects in metallic foams, Cosserat crystal plasticity, micromorphic crystal cleavage fracture...
  (Forest et al., 2000; Dillard et al., 2006; Zeghadi et al., 2007)

Aifantis E.C. (1987).
The physics of plastic deformation.
International Journal of Plasticity, vol. 3, pp 211–248.

Cahn J.W. and Hilliard J.E. (1958).
Free energy of a nonuniform system. I. Interfacial free energy.
The Journal of Chemical Physics, vol. 28, pp 258–267.

Coleman B.D. and Noll W. (1963).
The thermodynamics of elastic materials with heat conduction and viscosity.
Arch. Rational Mech. and Anal., vol. 13, pp 167–178.

Dillard T., Forest S., and Ienny P. (2006).
Micromorphic continuum modelling of the deformation and fracture behaviour of nickel foams.
European Journal of Mechanics A/Solids, vol. 25, pp 526–549.

Engelen R.A.B., Geers M.G.D., and Baaijens F.P.T. (2003).
Nonlocal implicit gradient-enhanced elasto-plasticity for the modelling of softening behaviour.
International Journal of Plasticity, vol. 19, pp 403–433.

Eringen A.C. and Suhubi E.S. (1964).
Nonlinear theory of simple microelastic solids.
Int. J. Engng Sci., vol. 2, pp 189–203, 389–404.

Fleck N.A. and Hutchinson J.W. (2001).
A reformulation of strain gradient plasticity.
Journal of the Mechanics and Physics of Solids, vol. 49, pp 2245–2271.

Forest S., Barbe F., and Cailletaud G. (2000).
Cosserat Modelling of Size Effects in the Mechanical Behaviour of Polycrystals and Multiphase Materials.
International Journal of Solids and Structures, vol. 37, pp 7105–7126.

Forest S. and Sievert R. (2003).
Elastoviscoplastic constitutive frameworks for generalized continua.
Acta Mechanica, vol. 160, pp 71–111.

Forest S. and Sievert R. (2006).
Nonlinear microstrain theories.
International Journal of Solids and Structures, vol. 43, pp 7224–7245.

Gurtin M.E. (1996).

Generalized Ginzburg–Landau and Cahn–Hilliard equations based on a microforce balance.
Physica D, vol. 92, pp 178–192.

Gurtin M.E. (2003).
On a framework for small–deformation viscoplasticity: free energy, microforces, strain gradients.
International Journal of Plasticity, vol. 19, pp 47–90.

Liebe T., Steinmann P., and Benallal A. (2001).
Theoretical and computational aspects of a thermodynamically consistent framework for geometrically linear gradient damage.
Comp. Methods Appli. Mech. Engng, vol. 190, pp 6555–6576.

Mindlin R.D. (1964).
Micro–structure in linear elasticity.
Arch. Rat. Mech. Anal., vol. 16, pp 51–78.

Mindlin R.D. (1965).
Second gradient of strain and surface–tension in linear elasticity.
Int. J. Solids Structures, vol. 1, pp 417–438.

Mindlin R.D. and Eshel N.N. (1968).
On first strain gradient theories in linear elasticity.
Int. J. Solids Structures, vol. 4, pp 109–124.

Mühlhaus H.B. and Aifantis E.C. (1991).
A variational principle for gradient plasticity.
Int. J. Solids Structures, vol. 28, pp 845–857.

Ubachs R.L.J.M., Schreurs P.J.G., and Geers M.G.D. (2004).
A nonlocal diffuse interface model for microstructure evolution of tin–lead solder.
Journal of the Mechanics and Physics of Solids, vol. 52, pp 1763–1792.

Zeghadi A., Nguyen F., Forest S., Gourgues A.-F., and Bouaziz O. (2007).
Ensemble averaging stress–strain fields in polycrystalline aggregates with a constrained surface microstructure–Part 1: Anisotropic elastic behaviour.
Philosophical Magazine, vol. 87, pp 1401–1424.