Shearing of a two–phase laminate according to strain gradient and micromorphic plasticity

Samuel Forest

Mines ParisTech / CNRS
Centre des Matériaux/UMR 7633
BP 87, 91003 Evry, France
Samuel.Forest@mines-paristech.fr

CENTRE DES MATERIAUX
P.M FOURT

Plan

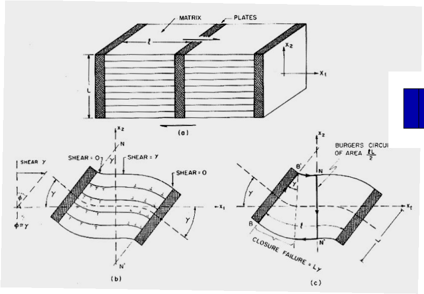
1. Shearing of a laminate for a strain gradient plasticity material
2. Shearing of a laminate for a micromorphic material

Plan

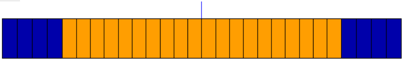
1. Shearing of a laminate for a strain gradient plasticity material

2. Shearing of a laminate for a micromorphic material

Confined plasticity



[Ashby, 1970]

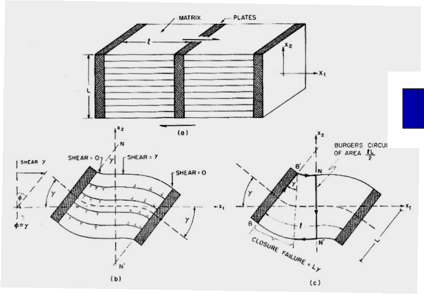


start animate end

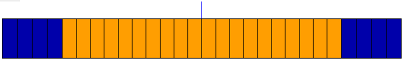
periodic simple shear test:
classical solution

classical continuum crystal plasticity cannot account for lattice curvature close to the interface (boundary layer effect)

Confined plasticity



[Ashby, 1970]

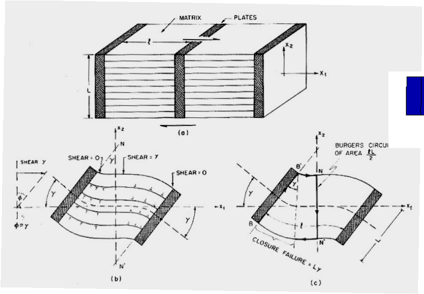


start animate end

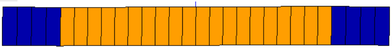
periodic simple shear test:
classical solution

classical continuum crystal plasticity cannot account for lattice curvature close to the interface (boundary layer effect)

Confined plasticity



[Ashby, 1970]

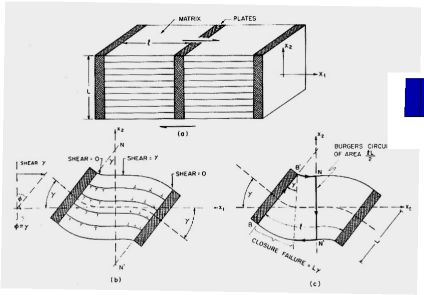


start animate end

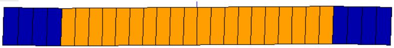
periodic simple shear test:
classical solution

classical continuum crystal plasticity cannot account for lattice curvature close to the interface (boundary layer effect)

Confined plasticity



[Ashby, 1970]

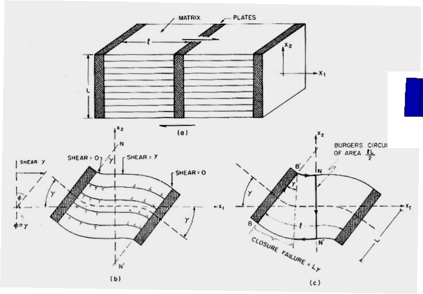


start animate end

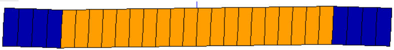
periodic simple shear test:
classical solution

classical continuum crystal plasticity cannot account for lattice curvature close to the interface (boundary layer effect)

Confined plasticity



[Ashby, 1970]

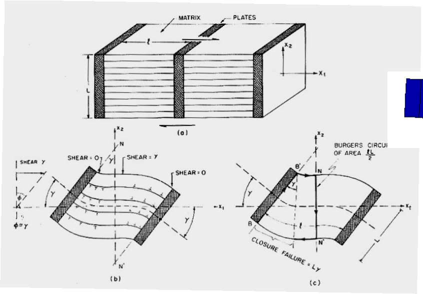


start animate end

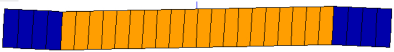
periodic simple shear test:
classical solution

classical continuum crystal plasticity cannot account for lattice curvature close to the interface (boundary layer effect)

Confined plasticity



[Ashby, 1970]

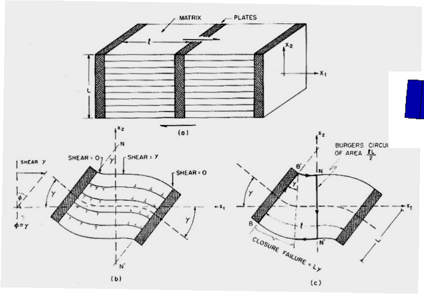


start animate end

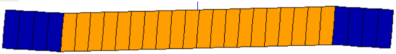
periodic simple shear test:
classical solution

classical continuum crystal plasticity cannot account for lattice curvature close to the interface (boundary layer effect)

Confined plasticity



[Ashby, 1970]

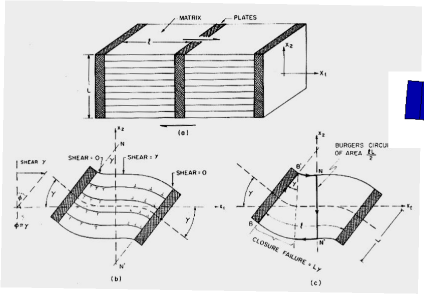


start animate end

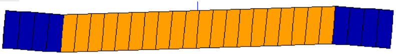
periodic simple shear test:
classical solution

classical continuum crystal plasticity cannot account for lattice curvature close to the interface (boundary layer effect)

Confined plasticity



[Ashby, 1970]

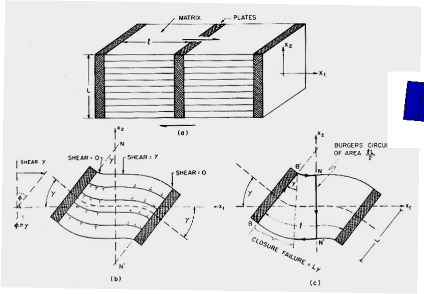


start animate end

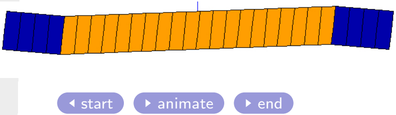
periodic simple shear test:
classical solution

classical continuum crystal plasticity cannot account for lattice curvature close to the interface (boundary layer effect)

Confined plasticity



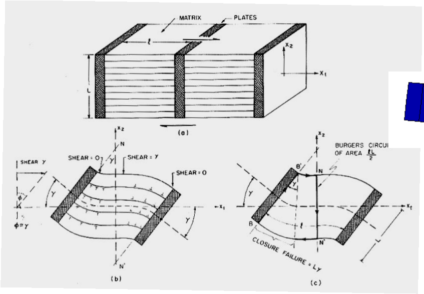
[Ashby, 1970]



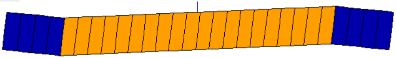
periodic simple shear test:
classical solution

classical continuum crystal plasticity cannot account for lattice curvature close to the interface (boundary layer effect)

Confined plasticity



[Ashby, 1970]

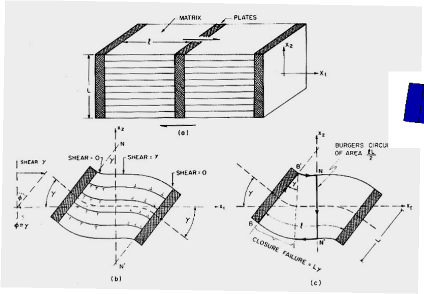


start animate end

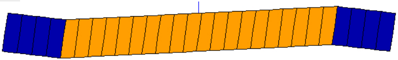
periodic simple shear test:
classical solution

classical continuum crystal plasticity cannot account for lattice curvature close to the interface (boundary layer effect)

Confined plasticity



[Ashby, 1970]

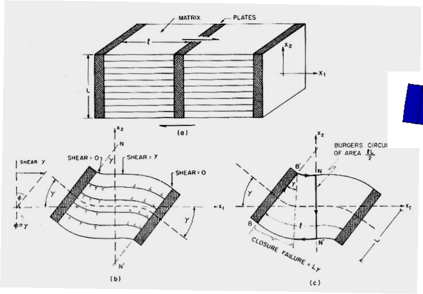


start animate end

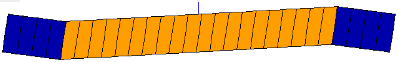
periodic simple shear test:
classical solution

classical continuum crystal plasticity cannot account for lattice curvature close to the interface (boundary layer effect)

Confined plasticity



[Ashby, 1970]

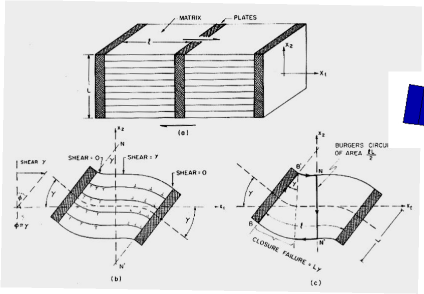


start animate end

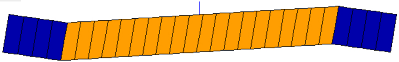
periodic simple shear test:
classical solution

classical continuum crystal plasticity cannot account for lattice curvature close to the interface (boundary layer effect)

Confined plasticity



[Ashby, 1970]

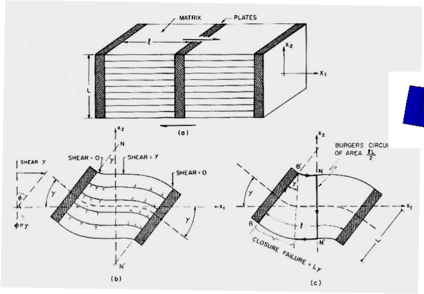


start animate end

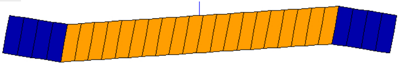
periodic simple shear test:
classical solution

classical continuum crystal plasticity cannot account for lattice curvature close to the interface (boundary layer effect)

Confined plasticity



[Ashby, 1970]

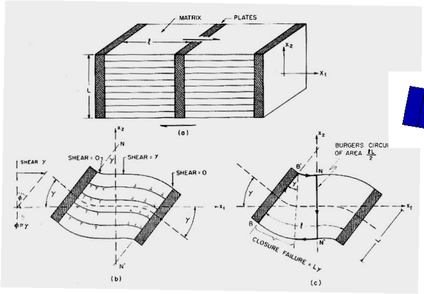


start animate end

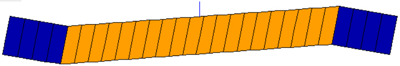
periodic simple shear test:
classical solution

classical continuum crystal plasticity cannot account for lattice curvature close to the interface (boundary layer effect)

Confined plasticity



[Ashby, 1970]

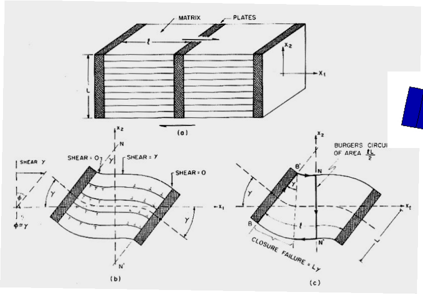


start animate end

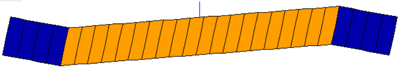
periodic simple shear test:
classical solution

classical continuum crystal plasticity cannot account for lattice curvature close to the interface (boundary layer effect)

Confined plasticity



[Ashby, 1970]

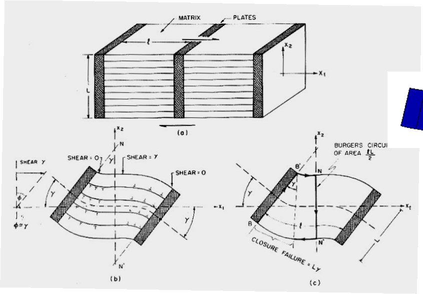


start animate end

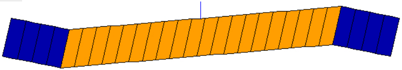
periodic simple shear test:
classical solution

classical continuum crystal plasticity cannot account for lattice curvature close to the interface (boundary layer effect)

Confined plasticity



[Ashby, 1970]

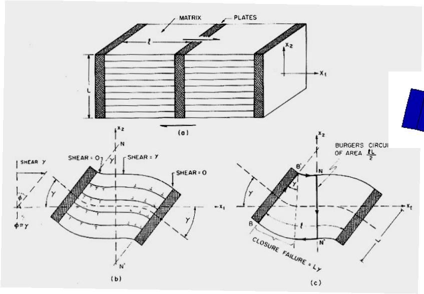


start animate end

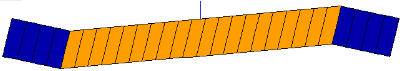
periodic simple shear test:
classical solution

classical continuum crystal plasticity cannot account for lattice curvature close to the interface (boundary layer effect)

Confined plasticity



[Ashby, 1970]

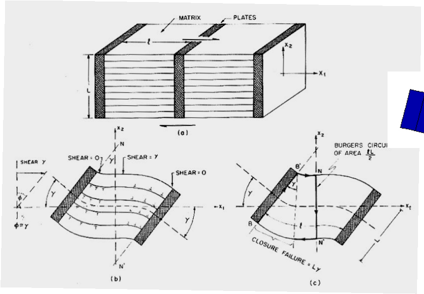


start animate end

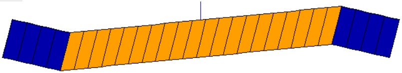
periodic simple shear test:
classical solution

classical continuum crystal plasticity cannot account for lattice curvature close to the interface (boundary layer effect)

Confined plasticity



[Ashby, 1970]

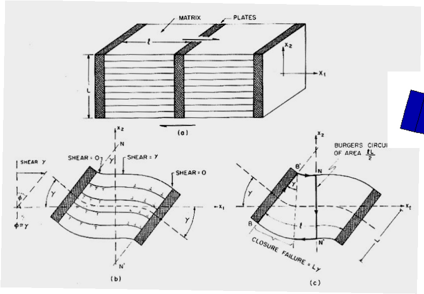


start animate end

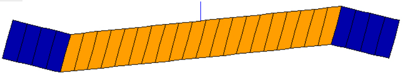
periodic simple shear test:
classical solution

classical continuum crystal plasticity cannot account for lattice curvature close to the interface (boundary layer effect)

Confined plasticity



[Ashby, 1970]

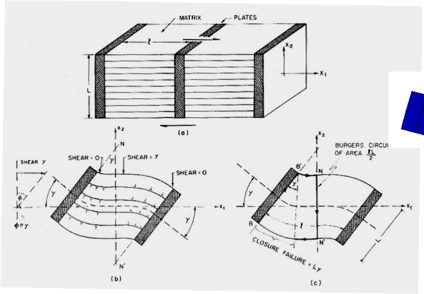


start animate end

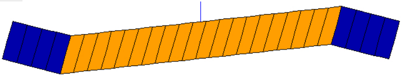
periodic simple shear test:
classical solution

classical continuum crystal plasticity cannot account for lattice curvature close to the interface (boundary layer effect)

Confined plasticity



[Ashby, 1970]

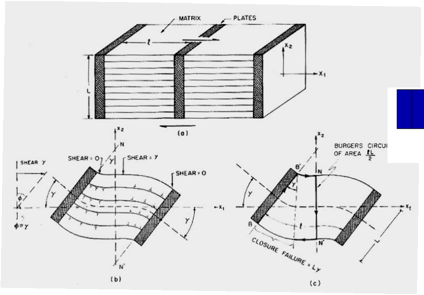


start animate end

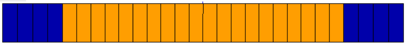
periodic simple shear test: classical solution

classical continuum crystal plasticity cannot account for lattice curvature close to the interface (boundary layer effect)

Confined plasticity



[Ashby, 1970]

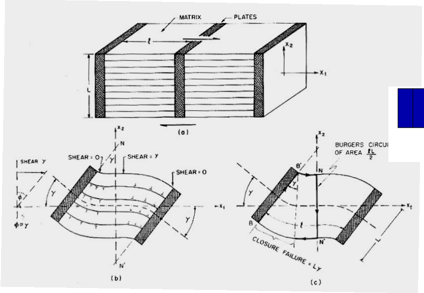


start animate end

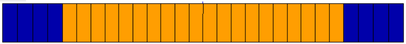
periodic shear test

Role of continuity requirements at the interface: displacement, lattice rotation, stress and generalized traction vectors

Confined plasticity



[Ashby, 1970]

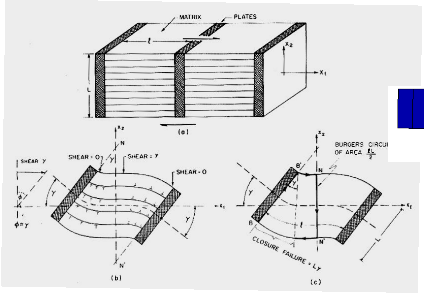


start animate end

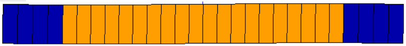
periodic shear test

Role of continuity requirements at the interface: displacement, lattice rotation, stress and generalized traction vectors

Confined plasticity



[Ashby, 1970]

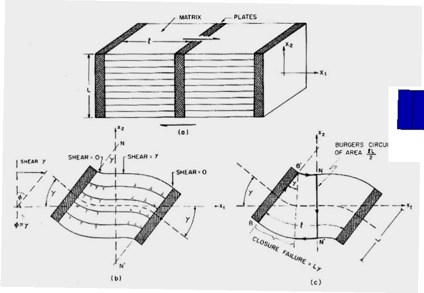


start animate end

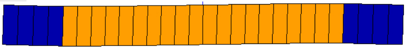
periodic shear test

Role of continuity requirements at the interface: displacement, lattice rotation, stress and generalized traction vectors

Confined plasticity



[Ashby, 1970]

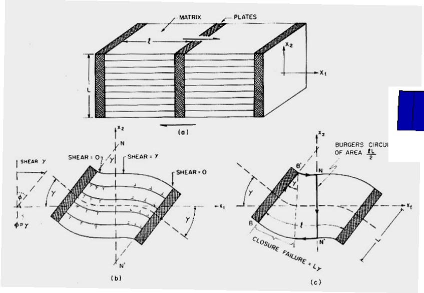


start animate end

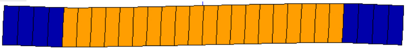
periodic shear test

Role of continuity requirements at the interface: displacement, lattice rotation, stress and generalized traction vectors

Confined plasticity



[Ashby, 1970]

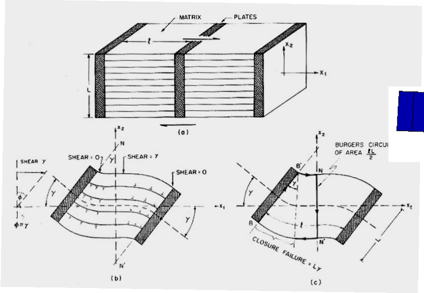


◂ start ▸ animate ▸ end

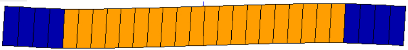
periodic shear test

Role of continuity requirements at the interface: displacement, lattice rotation, stress and generalized traction vectors

Confined plasticity



[Ashby, 1970]

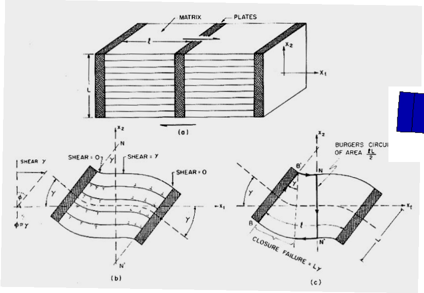


start animate end

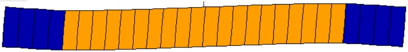
periodic shear test

Role of continuity requirements at the interface: displacement, lattice rotation, stress and generalized traction vectors

Confined plasticity



[Ashby, 1970]

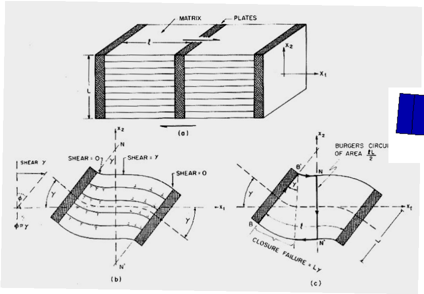


start animate end

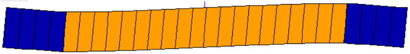
periodic shear test

Role of continuity requirements at the interface: displacement, lattice rotation, stress and generalized traction vectors

Confined plasticity



[Ashby, 1970]

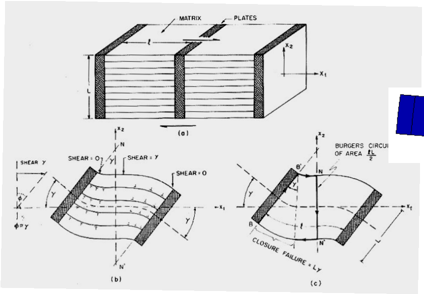


start animate end

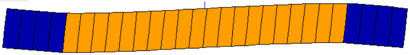
periodic shear test

Role of continuity requirements at the interface: displacement, lattice rotation, stress and generalized traction vectors

Confined plasticity



[Ashby, 1970]

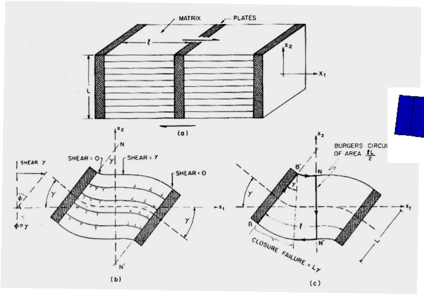


start animate end

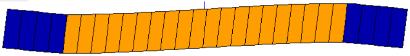
periodic shear test

Role of continuity requirements at the interface: displacement, lattice rotation, stress and generalized traction vectors

Confined plasticity



[Ashby, 1970]

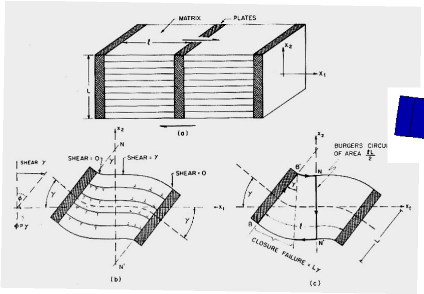


start animate end

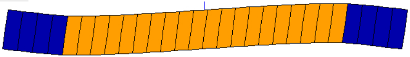
periodic shear test

Role of continuity requirements at the interface: displacement, lattice rotation, stress and generalized traction vectors

Confined plasticity



[Ashby, 1970]

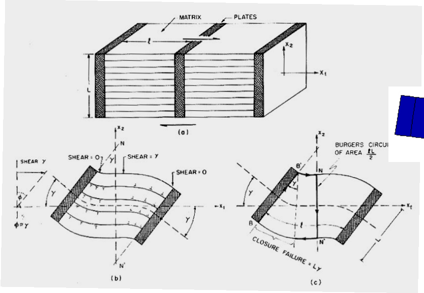


start animate end

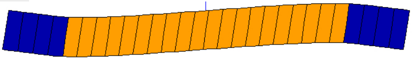
periodic shear test

Role of continuity requirements at the interface: displacement, lattice rotation, stress and generalized traction vectors

Confined plasticity



[Ashby, 1970]

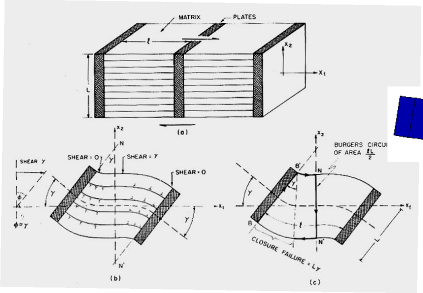


start animate end

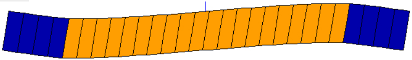
periodic shear test

Role of continuity requirements at the interface: displacement, lattice rotation, stress and generalized traction vectors

Confined plasticity



[Ashby, 1970]

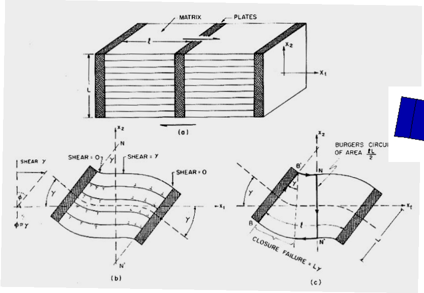


start animate end

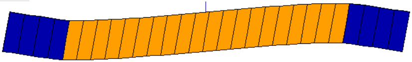
periodic shear test

Role of continuity requirements at the interface: displacement, lattice rotation, stress and generalized traction vectors

Confined plasticity



[Ashby, 1970]

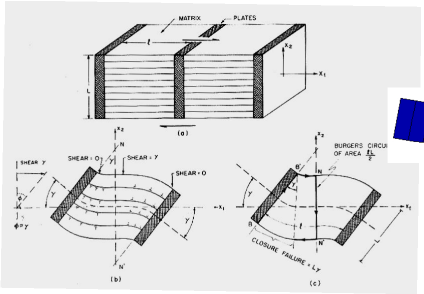


start animate end

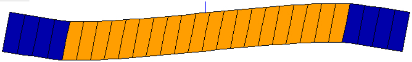
periodic shear test

Role of continuity requirements at the interface: displacement, lattice rotation, stress and generalized traction vectors

Confined plasticity



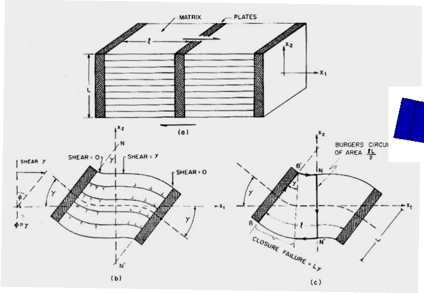
[Ashby, 1970]



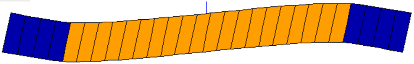
periodic shear test

Role of continuity requirements at the interface: displacement, lattice rotation, stress and generalized traction vectors

Confined plasticity



[Ashby, 1970]

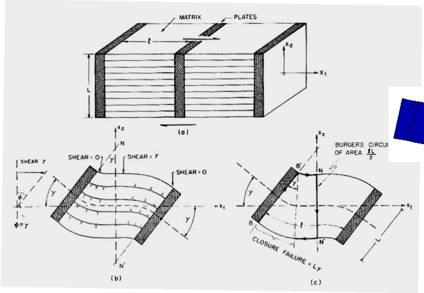


start animate end

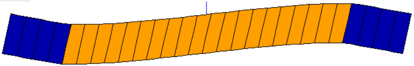
periodic shear test

Role of continuity requirements at the interface: displacement, lattice rotation, stress and generalized traction vectors

Confined plasticity



[Ashby, 1970]

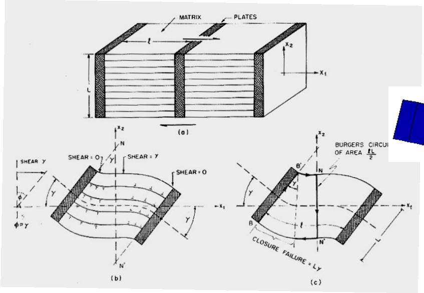


start animate end

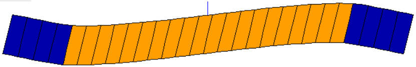
periodic shear test

Role of continuity requirements at the interface: displacement, lattice rotation, stress and generalized traction vectors

Confined plasticity



[Ashby, 1970]

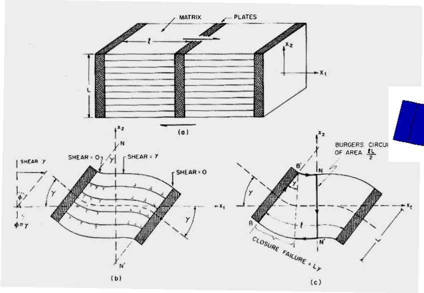


start animate end

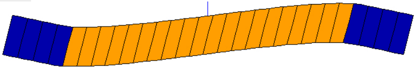
periodic shear test

Role of continuity requirements at the interface: displacement, lattice rotation, stress and generalized traction vectors

Confined plasticity



[Ashby, 1970]

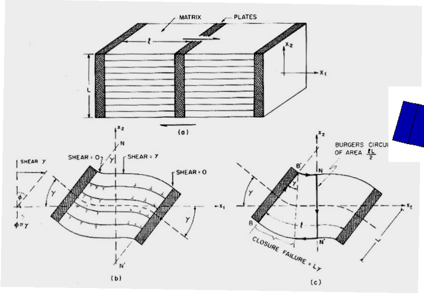


start animate end

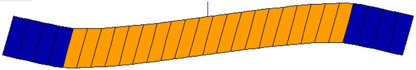
periodic shear test

Role of continuity requirements at the interface: displacement, lattice rotation, stress and generalized traction vectors

Confined plasticity



[Ashby, 1970]

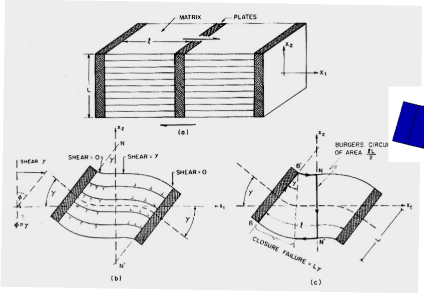


start animate end

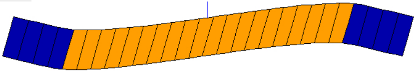
periodic shear test

Role of continuity requirements at the interface: displacement, lattice rotation, stress and generalized traction vectors

Confined plasticity



[Ashby, 1970]



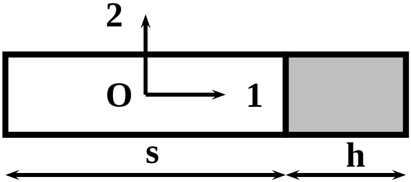
start animate end

periodic shear test

Role of continuity requirements at the interface: displacement, lattice rotation, stress and generalized traction vectors

Laminate microstructure under shear

Unit cell of a periodic two–phase laminate $\ell = s + h$



Aifantis material in the white (soft) phase, purely elastic gray (hard) phase

- Form of the solution for imposed mean shear $\bar{\gamma}$

$$u_1 = \bar{\gamma}\ x_2, \quad u_2(x_1) = u(x_1), \quad u_3 = 0$$

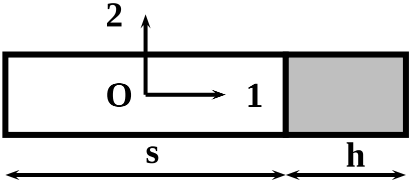
unknown periodic functions $u(x_1), p(x_1)$

- Deformation gradient and strain

$$[\nabla \underline{\mathbf{u}}] = \begin{bmatrix} 0 & \bar{\gamma} & 0 \\ u_{,1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [\underset{\sim}{\varepsilon}] = \begin{bmatrix} 0 & \frac{1}{2}(\bar{\gamma} + u_{,1}) & 0 \\ \frac{1}{2}(\bar{\gamma} + u_{,1}) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Laminate microstructure under shear

Unit cell of a periodic two–phase laminate $\ell = s + h$



Aifantis material in the white (soft) phase, purely elastic gray (hard) phase

- Form of the solution for imposed mean shear $\bar{\gamma}$

$$u_1 = \bar{\gamma}\, x_2, \quad u_2(x_1) = u(x_1), \quad u_3 = 0$$

  unknown periodic functions $u(x_1), p(x_1)$

- Deformation gradient and strain

$$[\nabla \underline{\mathbf{u}}] = \begin{bmatrix} 0 & \bar{\gamma} & 0 \\ u_{,1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [\underset{\sim}{\varepsilon}] = \begin{bmatrix} 0 & \frac{1}{2}(\bar{\gamma} + u_{,1}) & 0 \\ \frac{1}{2}(\bar{\gamma} + u_{,1}) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Resolution of the b.v.p.

Let us consider homogeneous isotropic elasticity and no hardening in the plastic phase for simplicity

- Equilibrium: homogeneous shear stress σ_{12} throughout the laminate
- Displacement in the hard phase

$$\sigma_{12} = \mu(\bar{\gamma} + u^h_{,1}) \quad \Longrightarrow \quad u^h_{,1} = C, \quad u^h = C x_1 + D$$

- Plastic strain in the soft phase

$$\underset{\sim}{\dot{\varepsilon}}^p = \frac{3}{2}\dot{p}\frac{\underset{\sim}{\mathbf{s}}}{J_2(\underset{\sim}{\sigma})}, \quad \underset{\sim}{\dot{\varepsilon}}^p = \frac{\sqrt{3}}{2}\dot{p}(\underline{\mathbf{e}}_1 \otimes \underline{\mathbf{e}}_2 + \underline{\mathbf{e}}_2 \otimes \underline{\mathbf{e}}_1)$$

from the yield condition we get

$$\sqrt{3}\sigma_{12} = R_0 - c p_{,11} \Longrightarrow p_{,111} = 0$$

so that the plastic strain is parabolic

$$p = \alpha(x_1^2 - \frac{s^2}{4})$$

- Continuity of plastic strain at the interface $\qquad p(\pm s/2) = 0$

Resolution of the b.v.p.

Let us consider homogeneous isotropic elasticity and no hardening in the plastic phase for simplicity

- Equilibrium: homogeneous shear stress σ_{12} throughout the laminate
- Displacement in the hard phase

$$\sigma_{12} = \mu(\bar{\gamma} + u^h_{,1}) \quad \Longrightarrow \quad u^h_{,1} = C, \quad u^h = C x_1 + D$$

- Plastic strain in the soft phase

$$\dot{\underset{\sim}{\varepsilon}}^p = \frac{3}{2}\dot{p}\frac{\underset{\sim}{s}}{J_2(\underset{\sim}{\sigma})}, \quad \dot{\underset{\sim}{\varepsilon}}^p = \frac{\sqrt{3}}{2}\dot{p}(\underline{e}_1 \otimes \underline{e}_2 + \underline{e}_2 \otimes \underline{e}_1)$$

from the yield condition we get

$$\sqrt{3}\sigma_{12} = R_0 - c p_{,11} \Longrightarrow p_{,111} = 0$$

so that the plastic strain is parabolic

$$p = \alpha(x_1^2 - \frac{s^2}{4})$$

- Continuity of plastic strain at the interface $\quad p(\pm s/2) = 0$

Resolution of the b.v.p.

Let us consider homogeneous isotropic elasticity and no hardening in the plastic phase for simplicity

- Equilibrium: homogeneous shear stress σ_{12} throughout the laminate
- Displacement in the hard phase

$$\sigma_{12} = \mu(\bar{\gamma} + u^h_{,1}) \quad \Longrightarrow \quad u^h_{,1} = C, \quad u^h = C x_1 + D$$

- Plastic strain in the soft phase

$$\dot{\underset{\sim}{\varepsilon}}^p = \frac{3}{2}\dot{p}\frac{\underset{\sim}{\mathbf{s}}}{J_2(\underset{\sim}{\sigma})}, \quad \dot{\underset{\sim}{\varepsilon}}^p = \frac{\sqrt{3}}{2}\dot{p}(\underline{\mathbf{e}}_1 \otimes \underline{\mathbf{e}}_2 + \underline{\mathbf{e}}_2 \otimes \underline{\mathbf{e}}_1)$$

  from the yield condition we get

$$\sqrt{3}\sigma_{12} = R_0 - c p_{,11} \Longrightarrow p_{,111} = 0$$

  so that the plastic strain is parabolic

$$p = \alpha(x_1^2 - \frac{s^2}{4})$$

- Continuity of plastic strain at the interface $\qquad p(\pm s/2) = 0$

Resolution of the b.v.p.

- Displacement in the soft phase

$$\sigma_{12} = \mu(\bar{\gamma} + u^s_{,1} - \sqrt{3}p) \quad \Longrightarrow \quad u^s_{,1} = C + \sqrt{3}p$$

$$u^s = (C - \alpha\sqrt{3}\frac{s^2}{4})x_1 + \alpha\frac{\sqrt{3}}{3}x_1^3$$

Interface conditions

- Displacement continuity at $x_1 = \pm s/2$

$$u^s(\frac{s}{2}) = u^h(\frac{s}{2}) \quad \Longrightarrow \quad -\sqrt{3}\alpha\frac{s^3}{12} = D$$

- Displacement periodicity at $x_1 = -s/2$ and $x_1 = s/2 + h$

$$u^s(-\frac{s}{2}) = u^h(\frac{s}{2} + h) \quad \Longrightarrow \quad \sqrt{3}\alpha\frac{s^3}{12} = C\ell + D$$

- Continuity of the stress vector at $x_1 = \pm s/2$

$$R_0 - 2c\alpha = \mu\sqrt{3}(\bar{\gamma} + C)$$

- The wanted constants are deduced from the previous equations

$$C = \frac{R_0 - \sqrt{3}\mu\bar{\gamma}}{\sqrt{3}\mu + \frac{12cl}{\sqrt{3}s^3}}, \quad D = -C\frac{\ell}{2}, \quad \alpha = -\frac{12}{\sqrt{3}}\frac{D}{s^3}$$

Interface conditions

- Displacement continuity at $x_1 = \pm s/2$

$$u^s(\frac{s}{2}) = u^h(\frac{s}{2}) \quad \Longrightarrow \quad -\sqrt{3}\alpha\frac{s^3}{12} = D$$

- Displacement periodicity at $x_1 = -s/2$ and $x_1 = s/2 + h$

$$u^s(-\frac{s}{2}) = u^h(\frac{s}{2} + h) \quad \Longrightarrow \quad \sqrt{3}\alpha\frac{s^3}{12} = C\ell + D$$

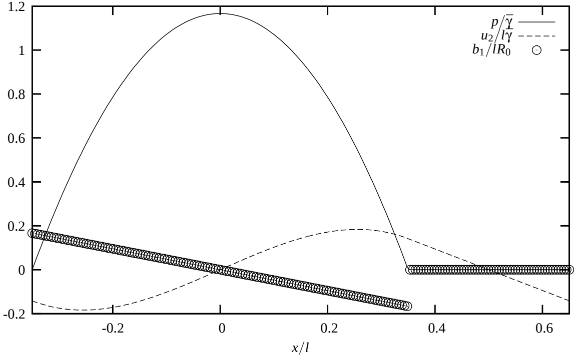
- Continuity of the stress vector at $x_1 = \pm s/2$

$$R_0 - 2c\alpha = \mu\sqrt{3}(\bar{\gamma} + C)$$

- The wanted constants are deduced from the previous equations

$$C = \frac{R_0 - \sqrt{3}\mu\bar{\gamma}}{\sqrt{3}\mu + \frac{12cl}{\sqrt{3}s^3}}, \quad D = -C\frac{\ell}{2}, \quad \alpha = -\frac{12}{\sqrt{3}}\frac{D}{s^3}$$

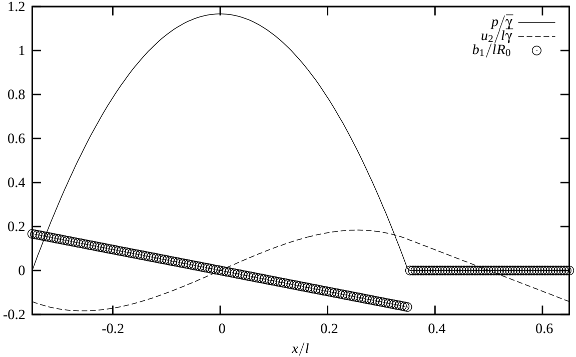
Plastic strain profile in the channel



μ (MPa)	R_0 (MPa)	c (MPa.mm^2)	f	ℓ (μm)	$\bar{\gamma}$
30000	20	0.005	0.7	10	0.01

- Characteristic length: $\ell_c = \sqrt{c/\mu} = 0.4\ \mu$m, leading to strong size effects in the micron range and below

Plastic strain profile in the channel



μ (MPa)	R_0 (MPa)	c (MPa.mm^2)	f	ℓ (μm)	$\bar{\gamma}$
30000	20	0.005	0.7	10	0.01

- The higher order stress $b_1 = 2c\alpha$ experiences a jump at the interface $s = \pm s/2$:

$$b_1(\frac{s^+}{2}) - b_1(\frac{s^-}{2}) = 0 - c\alpha s, \quad [\![b_1]\!](\frac{s}{2}) = -c\alpha s$$

Overall size effect

- Macroscopic stress strain relation

$$\frac{\sigma_{12}}{\mu} = \frac{1}{\mu f s^2 + 4c}\left(\frac{\sqrt{3}}{3} f s^2 R_0 + 4c\bar{\gamma}\right)$$

bilinear response depending explicitly on channel size s

- Macroscopic stress vs mean plastic strain;

$$\bar{p} = \frac{1}{\ell}\int_{-s/2}^{s/2} p(x_1)\, dx_1 \quad \Longrightarrow \quad \sqrt{3}\bar{p} = f\bar{\gamma} - C(1-f) - f\frac{\sigma_{12}}{\mu}$$

$$\sigma_{12} = \frac{R_0}{\sqrt{3}} + \frac{4\sqrt{3}c}{f^3\ell^2}\bar{p}$$

microstructure–induced linear hardening depending on unit cell size ℓ

- Limit cases
  - ⋆ thick channels: size independent threshold $\sigma_{12} = R_0/\sqrt{3}$
  - ⋆ thin films: scaling law $\sigma_{12}/\bar{p} \sim 1/\ell^2$

Overall size effect

- Macroscopic stress strain relation

$$\frac{\sigma_{12}}{\mu} = \frac{1}{\mu f s^2 + 4c}\left(\frac{\sqrt{3}}{3} f s^2 R_0 + 4c\bar{\gamma}\right)$$

bilinear response depending explicitly on channel size s

- Macroscopic stress vs mean plastic strain;

$$\bar{p} = \frac{1}{\ell}\int_{-s/2}^{s/2} p(x_1)\, dx_1 \quad \Longrightarrow \quad \sqrt{3}\bar{p} = f\bar{\gamma} - C(1-f) - f\frac{\sigma_{12}}{\mu}$$

$$\sigma_{12} = \frac{R_0}{\sqrt{3}} + \frac{4\sqrt{3}c}{f^3\ell^2}\bar{p}$$

microstructure–induced linear hardening depending on unit cell size ℓ

- Limit cases
  - ⋆ thick channels: size independent threshold $\sigma_{12} = R_0/\sqrt{3}$
  - ⋆ thin films: scaling law $\sigma_{12}/\bar{p} \sim 1/\ell^2$

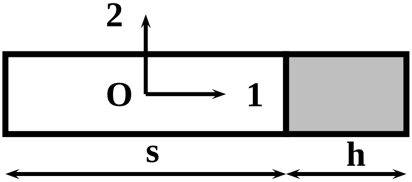
Plan

1. Shearing of a laminate for a strain gradient plasticity material

2. Shearing of a laminate for a micromorphic material

Laminate microstructure under shear

Unit cell of a periodic two–phase laminate $\ell = s + h$



Micromorphic material in the white (soft) phase, purely elastic micromorphic gray (hard) phase

- Form of the solution for impose mean shear $\bar{\gamma}$

$$u_1 = \bar{\gamma}\, x_2, \quad u_2(x_1) = u(x_1), \quad u_3 = 0$$

  unknown periodic functions $u(x_1), p(x_1), p_\chi(x_1)$

- Deformation gradient and strain

$$[\nabla \underline{\mathbf{u}}] = \begin{bmatrix} 0 & \bar{\gamma} & 0 \\ u_{,1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [\underset{\sim}{\varepsilon}] = \begin{bmatrix} 0 & \frac{1}{2}(\bar{\gamma} + u_{,1}) & 0 \\ \frac{1}{2}(\bar{\gamma} + u_{,1}) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Resolution of the b.v.p.

Let us consider homogeneous isotropic elasticity, homogeneous H_χ and no hardening in the plastic phase for simplicity

- The shear stress is uniform throughout the laminate and takes the value

$$\sqrt{3}\sigma_{12} = R_0 + R = R_0 + H_\chi(p - p_\chi) = R_0 - Ap_{\chi,11}$$

- Derivation of the previous equations with respect to x_1 shows that $p_{\chi,111} = 0$ which leads to the parabolic profile of the micro–plastic deformation in the soft phase

$$p_\chi(x) = \alpha x^2 + \beta, \quad \forall |x| \leq \frac{s}{2}$$

  Note that

$$\sqrt{3}\sigma_{12} = R_0 - 2A\alpha$$

- The parabolic plastic strain profile follows

$$p = \alpha x^2 + \beta - \frac{2A}{H_\chi}\alpha$$

Resolution of the b.v.p.

A new feature of the model is that the microplastic strain p_χ does not vanish in general in the hard phase, whereas p does:

$$p_\chi - \frac{A^h}{H_\chi} \Delta p_\chi = 0$$

$$p_\chi^h = \alpha_h \cosh \omega_h (x - \frac{l}{2}), \quad \frac{s}{2} \leq x \leq \frac{s}{2} + h, \quad \text{with} \quad \omega_h^2 = \frac{H_\chi}{A_h}$$

the p_χ^h profile is of hyperbolic nature

Interface conditions

- Continuity of micro–plastic deformation at $x = s/2$:

$$\alpha\frac{s^2}{4} + \beta = \alpha_h \cosh \omega_h \frac{h}{2}$$

- Continuity of the generalized stress component b_1:

$$A\alpha s = -A_h\alpha_h\omega_h \sinh \omega_h \frac{h}{2}$$

Interface conditions

The displacement in the plastic and elastic phases can be expressed as

$$u^s = \alpha \frac{x^3}{\sqrt{3}} + \left(\sqrt{3}\beta - \bar{\gamma} + \frac{R_0}{\sqrt{3}\mu} - 2A\alpha \left(\frac{1}{\sqrt{3}\mu} + \frac{\sqrt{3}}{H_\chi} \right) \right) x$$

$$u^h = \left(\frac{1}{\sqrt{3}\mu} (R_0 - 2A\alpha) - \bar{\gamma} \right) x + C$$

They are used to exploit two additional interface conditions

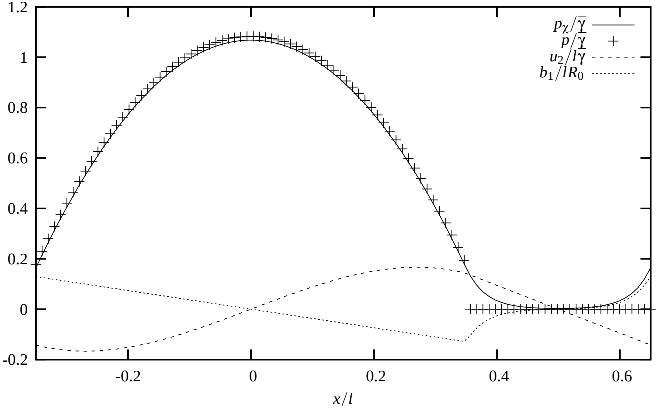
- Continuity of the displacement at $x = s/2$:
  $u^s(s/2) = u^h(s/2)$

$$\alpha \frac{s^3}{8\sqrt{3}\mu} + \sqrt{3}(\beta - \frac{2A\alpha}{H_\chi})\frac{s}{2} = C$$

- Periodicity of the displacement component
  $u^s(-s/2) = u^h(s/2 + h)$

$$-(\frac{\sigma_{12}}{\mu} - \bar{\gamma})\ell + \sqrt{3}(\beta + \frac{2A\alpha}{H_\chi})\frac{s}{2} - \alpha \frac{s^3}{8\sqrt{3}} = C$$

Plastic strain profiles in the channel



μ (MPa)	R_0 (MPa)	H_χ (MPa)	A (MPa.mm^2)	f	ℓ (μm)	$\bar{\gamma}$
30000	20	50000	0.005	0.7	10	0.01

Overall size effect

The scaling law results from the expression of the overall stress σ_{12} as a function of the mean plastic strain over the unit cell:

$$\bar{p} = \frac{1}{\ell}\int_{-\frac{s}{2}}^{\frac{s}{2}} (\alpha x^2 + \beta - \frac{2A\alpha}{H_\chi})\, dx = \beta f \left(1 - \frac{1}{L^2}\left(\frac{s^2}{12} - \frac{2A}{H_\chi}\right)\right)$$

with $L^2 = \frac{s^2}{4} + \frac{A}{A_h}\frac{s}{\omega_h}\mathrm{cotanh}(\omega_h \frac{h}{2}) = -\frac{\beta}{\alpha}$. The uniform stress component can now be expressed as a function of the volume fraction f of the soft phase and of the unit cell size l:

$$\sqrt{3}\sigma_{12} = R_0 + \frac{2A}{f}\frac{\bar{p}}{\frac{f^2\ell^2}{6} + \frac{2A}{H_\chi} + \frac{A}{A_h}\frac{f\ell}{\omega_h}\mathrm{cotanh}\left(\omega_h \frac{h}{2}\right)}$$

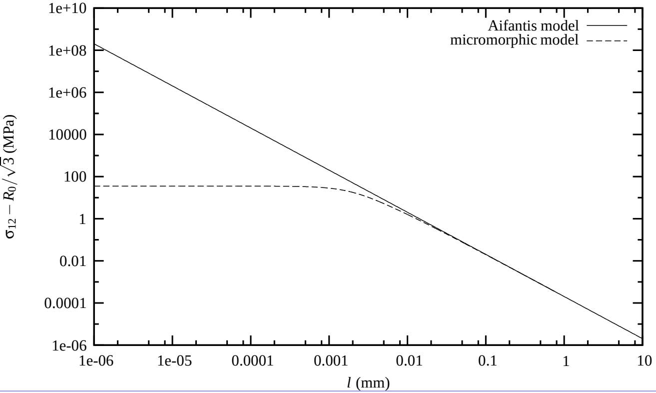
displaying a size–dependent overall linear hardening

Scaling laws

Two limit cases naturally arise

- Internal constraint $H_\chi \to \infty$ for which the strain gradient plasticity model is retrieved
- Unit cell size $\ell \to 0$ leads to saturation stress

$$\sqrt{3}\sigma_{12} - R_0 \sim H_\chi \frac{1-f}{f}\bar{p}$$



Acharya A. (2007).
Jump condition for GND evolution as a constraint on slip transmission at grain boundaries.
Philosophical Magazine, vol. 87, pp 1349–1359.

Anand L., Aslan O., and Chester S.A. (2012).
A large–deformation gradient theory for elastic–plastic materials: Strain softening and regularization of shear bands.
International Journal of Plasticity, vol. 30–31, pp 116–143.

Aslan O., Cordero N. M., Gaubert A., and Forest S. (2011).
Micromorphic approach to single crystal plasticity and damage.
International Journal of Engineering Science, vol. 49, pp 1311–1325.

Bargmann S., Ekh M., Runesson K., and Svendsen B. (2010).
Modeling of polycrystals with gradient crystal plasticity: A comparison of strategies.
Philosophical Magazine, vol. 90, pp 1263–1288.

Bayley C.J., Brekelmans W.A.M., and Geers M.G.D. (2007).

A three–dimensional dislocation field crystal plasticity approach applied to miniaturized structures.
Philosophical Magazine, vol. 87, pp 1361–1378.

Cermelli P. and Gurtin M.E. (2001).
On the characterization of geometrically necessary dislocations in finite plasticity.
Journal of the Mechanics and Physics of Solids, vol. 49, pp 1539–1568.

Conti S. and Ortiz M. (2005).
Dislocation Microstructures and the Effective Behavior of Single Crystals.
Arch. Rational Mech. Anal., vol. 176, pp 103–147.

Cordero N.M., Gaubert A., Forest S., Busso E., Gallerneau F., and Kruch S. (2010).
Size effects in generalised continuum crystal plasticity for two–phase laminates.

Journal of the Mechanics and Physics of Solids, vol. 58, pp 1963–1994.

Cordero N. M., Forest S., and Busso E. P. (2012).
Generalised continuum modelling of grain size effects in polycrystals.
Comptes Rendus Mécanique, vol. 340, pp 261–274.

Cottura M., Le Bouar Y., Finel A., Appolaire B., Ammar K., and Forest S. (2012).
A phase field model incorporating strain gradient viscoplasticity: application to rafting in Ni-base superalloys.
Journal of the Mechanics and Physics of Solids, vol. 60, pp 1243–1256.

Forest S. (2009).
The micromorphic approach for gradient elasticity, viscoplasticity and damage.
ASCE Journal of Engineering Mechanics, vol. 135, pp 117–131.

Forest S., Barbe F., and Cailletaud G. (2000).

Cosserat Modelling of Size Effects in the Mechanical Behaviour of Polycrystals and Multiphase Materials.
International Journal of Solids and Structures, vol. 37, pp 7105–7126.

Gurtin M.E. and Anand L. (2008).
Nanocrystalline grain boundaries that slip and separate: A gradient theory that accounts for grain-boundary stress and conditions at a triple-junction.
Journal of the Mechanics and Physics of Solids, vol. 56, pp 184–199.

Gurtin M.E. and Needleman A. (2005).
Boundary conditions in small–deformation, single–crystal plasticity that account for the Burgers vector.
Journal of the Mechanics and Physics of Solids, vol. 53, pp 1–31.

Neff P., Sydow A., and Wieners C. (2009).
Numerical approximation of incremental infinitesimal gradient plasticity.

International Journal for Numerical Methods in Engineering, vol. 77, pp 414–436.

Okumura D., Higashi Y., Sumida K., and Ohno N. (2007).
A homogenization theory of strain gradient single crystal plasticity and its finite element discretization.
International Journal of Plasticity, vol. 23, pp 1148–1166.

Svendsen B. (2002).
Continuum thermodynamic models for crystal plasticity including the effects of geometrically-necessary dislocations.
J. Mech. Phys. Solids, vol. 50, pp 1297–1329.

Wulfinghoff S. and Böhlke T. (2012).
Equivalent plastic strain gradient enhancement of single crystal plasticity: theory and numerics.
Proceedings of the Royal Society A, vol. 468, pp 2682–2703.