Simple glide problem for the Cosserat continuum

We study the problem of an infinite plate, of height H, fixed at the bottom and with prescribed microrotation or couple-stress vector at the top, as shown on figure 1. We assume plane strain

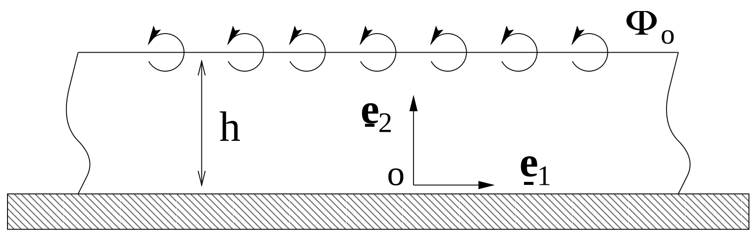

FIG. 1 – Simple glide problem for the Cosserat continuum.

conditions : $u_3 = 0$ and that
$\Phi_1 = \Phi_2 = 0$, and a displacement field and rotation field of the form :

$$\underline{\boldsymbol{u}} = \begin{pmatrix} u(y) \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \underline{\boldsymbol{\Phi}} = \begin{pmatrix} 0 \\ 0 \\ \Phi(y) \end{pmatrix}$$

The deformations associated with the kinematic fields defined above write :

$$\underset{\sim}{\boldsymbol{e}} = \underset{\sim}{\boldsymbol{\nabla}} \underline{\boldsymbol{u}} + \underset{\sim}{\boldsymbol{\epsilon}}.\underline{\boldsymbol{\Phi}} = \begin{pmatrix} 0 & u' + \Phi & 0 \\ -\Phi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\underset{\sim}{\boldsymbol{\kappa}} = \underline{\boldsymbol{\Phi}} \otimes \underline{\boldsymbol{\nabla}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \Phi' & 0 \end{pmatrix}$$

1 Linear elastic case

The elastic constitutive equations allow us to obtain the expression of the fields $\underset{\sim}{\boldsymbol{\sigma}}$ and $\underset{\sim}{\boldsymbol{M}}$:

$$\underset{\sim}{\boldsymbol{\sigma}} = \begin{bmatrix} 0 & \mu u' + \mu_c(u' + 2\Phi) & 0 \\ \mu u' - \mu_c(u' + 2\Phi) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$\underset{\sim}{\boldsymbol{M}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & (\beta - \gamma)\Phi' \\ 0 & (\beta + \gamma)\Phi' & 0 \end{bmatrix}$$

In general, we assume that the coefficients β and γ are equal so that $\mu_{23} = 0$ and $\mu_{32} = 2\beta\Phi'$. This hypothesis allows us to fulfill the plane strain conditions. On the other hand, we notice on these expressions the non-symmetry of the strain and stress tensors. The equilibrium equations write :

$$\left\{ \begin{array}{l} \underline{\mathbf{div}}\, \underset{\sim}{\boldsymbol{\sigma}} \ + \ \underline{\boldsymbol{f}} \ = \ \underline{\mathbf{0}} \\ \underline{\mathbf{div}}\, \underset{\sim}{\boldsymbol{M}} \ + \ 2\, \underline{\overset{\times}{\boldsymbol{\sigma}}} \ + \underline{\boldsymbol{c}} \ = \ \underline{\mathbf{0}} \end{array} \right. \quad \text{with} \quad \underline{\overset{\times}{\boldsymbol{\sigma}}} \ = \ \left[\begin{array}{c} 0 \\ 0 \\ -\mu_c(u' + 2\Phi) \end{array} \right]$$

with :

$$\left\{ \begin{array}{l} \sigma_{12,2} \ = \ 0 \\ \mu_{32,2} \ + \ 2\, \overset{\times}{\sigma}_3 \ = \ 0 \end{array} \right.$$

$$\left\{ \begin{array}{l} \sigma_{12} \ = \ \mu u' \ + \ \mu_c(u' + 2\Phi) \ = \ \text{Cst} \ = \ \sigma_0 \\ \\ 2\beta\Phi'' \ - \ 2\mu_c(u' + 2\Phi) \ = \ 0 \end{array} \right.$$

$$\left\{ \begin{array}{l} u' \ = \ \dfrac{\sigma_0 - 2\mu_c\Phi}{\mu + \mu_c} \\ \\ \beta(\mu + \mu_c)\Phi'' \ - \ 2\mu\mu_c\Phi \ = \ \mu_c\sigma_0 \end{array} \right.$$

The particular solution of the differential equation in Φ is $\Phi_1 \ = \ -\dfrac{\sigma_0}{2\mu}$.

The solution of the homogeneous system is : $\Phi_2 \ = \ A\, e^{y/l_c} \ + \ B\, e^{-y/l_c}$ with $\dfrac{1}{l_c} \ = \ \sqrt{\dfrac{2\mu\mu_c}{\beta(\mu + \mu_c)}}$.

We notice that l_c has the dimension of a length. It is the characteristic length of th problem. The general solution for the kinematic fields $\underline{\boldsymbol{u}}$ and $\underline{\boldsymbol{\Phi}}$ is :

$$\left\{ \begin{array}{l} \Phi \ = \ \Phi_1 + \Phi_2 \ = \ -\dfrac{\sigma_0}{2\mu} + A\, e^{y/l_c} \ + \ B\, e^{-y/l_c} \\ u \ = \ \dfrac{\sigma_0}{\mu} y \ - \ \dfrac{2\mu_c l_c}{(\mu + \mu_c)}(A\, e^{y/l_c} - B\, e^{-y/l_c}) \ + \ C \end{array} \right.$$

The constants obtained after integrating the equilibrium equations are determined from the boundary conditions. We note that, whatever the boundary conditions are, the solution has a hyperbolic form, which differs completely from the classical case of single slip where the deformations are homogeneous and the displacements are linear in y. We consider the case where we impose a microrotation Φ_0 at the upper surface, which is free from force stresses. The boundary conditions are then :

$$\left\{ \begin{array}{l} \bullet \ \text{lower surface} : u \ = \ \Phi \ = \ 0 \\ \\ \bullet \ \text{upper surface} : \left\| \begin{array}{l} \Phi \ = \ \Phi_0 \\ \\ \underline{\boldsymbol{e}}_1 \cdot \underset{\sim}{\boldsymbol{\sigma}} \cdot \underline{\boldsymbol{e}}_2 \ = \ \sigma_{12} \ = \ 0 \end{array} \right. \end{array} \right.$$

Hence

$$\begin{cases} \sigma_{12}(H) \;=\; 0 \;\;\Rightarrow\;\; \sigma_0 \;=\; 0 \\ \Phi(0) \;=\; 0 \;\;\Rightarrow\;\; A \;+\; B \;=\; 0 \\ \Phi(H) \;=\; \Phi_0 \;\;\Rightarrow\;\; A\, e^{H/l_c} \;+\; B\, e^{-H/l_c} \;=\; \Phi_0 \\ u(0) \;=\; 0 \;\;\Rightarrow\;\; -\dfrac{2\mu_c l_c}{(\mu+\mu_c)}\,(A-B) \;+\; C \;=\; 0 \end{cases}$$

The analytical solution for the elastic problem with imposed microrotation is written

$$\Phi(y) \;=\; \Phi_0\,\frac{\sinh(\mathrm{y/l_c})}{\sinh(\mathrm{H/l_c})}$$

$$u(y) \;=\; \frac{2\mu_c l_c \Phi_0}{(\mu+\mu_c)\sinh(\mathrm{H/l_c})}\,(1-\cosh(\mathrm{y/l_c}))$$

The solution shows that a boundary layer of size characterized by l_c exists at the top of the strip. This solution can be used to test numerical implementation of the Cosserat model.

The profiles of microrotation are drawn on figure 1 for the following values of the material parameters : $\mu = 30000$ MPa, $\beta = 500$ MPa.mm^2, $\mu_c = 100000$ MPa, $l_c/H \simeq 0.1$.

Limit cases can be worked out for the constrained Cosserat continuum (couple stress theory) when $\mu_c \to \infty$ [Koiter, 1963, Nowacki, 1986], and when the characteristic length l_c goes to zero, for which the Cauchy continuum is retrieved. The displacement vanishes in the latter case since the Cauchy continuum is not sensitive to an applied microrotation.

2 Elastoplastic case at small deformations

2.1 Constitutive equations

The first trials for an extension of classical von Mises elastoplasticity to the Cosserat continuum are due to [Sawczuk, 1967], [Lippmann, 1969], [Besdo, 1974], [Mühlhaus and Vardoulakis, 1987] and [Borst, 1991, Borst, 1993]. They belong to the class of single criterion plasticity models. The following form of the extended von Mises criterion was proposed :

$$f(\underset{\sim}{\boldsymbol{\sigma}}, \underset{\sim}{\boldsymbol{M}}, R) \;=\; J_2(\underset{\sim}{\boldsymbol{\sigma}}, \underset{\sim}{\boldsymbol{M}}) \;-\; R(p) \tag{1}$$

$$J_2(\underset{\sim}{\boldsymbol{\sigma}}, \underset{\sim}{\boldsymbol{M}}) = \sqrt{a\underset{\sim}{\boldsymbol{\sigma}}' : \underset{\sim}{\boldsymbol{\sigma}}' + b\underset{\sim}{\boldsymbol{M}} : \underset{\sim}{\boldsymbol{M}}} \tag{2}$$

where $\underset{\sim}{\boldsymbol{\sigma}}'$ is the deviatoric part of $\underset{\sim}{\boldsymbol{\sigma}}$, a, b are material parameters. The flow rules and plastic multiplier then read :

$$\dot{\underset{\sim}{\boldsymbol{e}}}^p = \dot{p}\,\frac{a\,\underset{\sim}{\boldsymbol{\sigma}}'}{J_2(\underset{\sim}{\boldsymbol{\sigma}}, \underset{\sim}{\boldsymbol{M}})}, \quad \dot{\underset{\sim}{\boldsymbol{\kappa}}}^p = \dot{p}\,\frac{b\,\underset{\sim}{\boldsymbol{M}}}{J_2(\underset{\sim}{\boldsymbol{\sigma}}, \underset{\sim}{\boldsymbol{M}})} \tag{3}$$

$$\dot{p} = \sqrt{\frac{1}{a}\dot{\underset{\sim}{\boldsymbol{e}}}^p : \dot{\underset{\sim}{\boldsymbol{e}}}^p + \frac{1}{b}\dot{\underset{\sim}{\boldsymbol{\kappa}}}^p : \dot{\underset{\sim}{\boldsymbol{\kappa}}}^p} \tag{4}$$

The use of the consistency condition $\dot{f} = 0$ under plastic loading yields the following expression of the plastic multiplier :

$$\dot{p} = \frac{\underset{\sim}{\boldsymbol{N}} : \underset{\approx}{\boldsymbol{E}} : \dot{\underset{\sim}{\boldsymbol{e}}} + \underset{\sim}{\boldsymbol{N}}_c : \underset{\approx}{\boldsymbol{C}} : \dot{\underset{\sim}{\boldsymbol{\kappa}}}}{H + \underset{\sim}{\boldsymbol{N}} : \underset{\approx}{\boldsymbol{E}} : \underset{\sim}{\boldsymbol{N}} + \underset{\sim}{\boldsymbol{N}}_c : \underset{\approx}{\boldsymbol{C}} : \underset{\sim}{\boldsymbol{N}}_c} \tag{5}$$

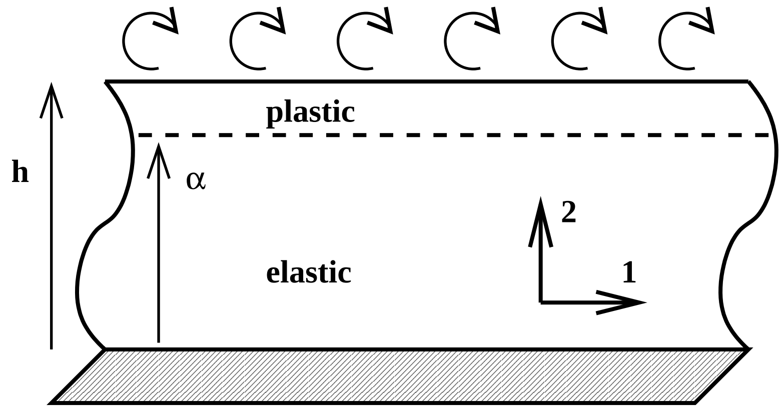

FIG. 2 – Simple glide test for a Cosserat infinite layer : elastic and elastoplastic domains, boundary conditions.

This expression involves the normal tensors $\underset{\sim}{\boldsymbol{N}}$ and $\underset{\sim}{\boldsymbol{N}}_c$ to the yield surface, the hardening modulus H and the tensors of elastic moduli $\underset{\approx}{\boldsymbol{E}}$ and $\underset{\approx}{\boldsymbol{C}}$ for linear elasticity (for a material admitting at least point symmetry) :

$$\underset{\sim}{\boldsymbol{N}} = \frac{\partial f}{\partial \underset{\sim}{\boldsymbol{\sigma}}}, \quad \underset{\sim}{\boldsymbol{N}}_c = \frac{\partial f}{\partial \underset{\sim}{\boldsymbol{M}}}, \quad H = \frac{\partial R}{\partial p}, \quad \underset{\approx}{\boldsymbol{E}} = \frac{\partial^2 \Psi}{\partial \underset{\sim}{\boldsymbol{e}}^e \, \partial \underset{\sim}{\boldsymbol{e}}^e}, \quad \underset{\approx}{\boldsymbol{C}} = \frac{\partial^2 \Psi}{\partial \underset{\sim}{\boldsymbol{\kappa}}^e \, \partial \underset{\sim}{\boldsymbol{\kappa}}^e} \tag{6}$$

The condition of plastic loading for the material point is that the numerator of equation (5) is positive, provided that the denominator remains positive, which still allows softening behaviour ($H < 0$).

2.2 Solution for the generalized von Mises model

It is important to see the respective role of Cosserat characteristic lengths appearing in the elastic and plastic constitutive equations in some simple situations. Analytical solutions for an isotropic elastic-ideally plastic Cosserat material involving one or two yield functions can be worked out in the case of the Cosserat glide. The considered boundary value problem is depicted on figures 2. The detailed solutions are provided below. Two characteristic lengths can be defined :

$$l_e = \sqrt{\frac{\beta}{\mu}}, \quad l_p = \sqrt{\frac{a}{b}} \tag{7}$$

(see equation (9) for the definition of isotropic Cosserat elastic bending modulus β). In the glide test, the material can be divided into elastic and plastic zones (figures 2). Characteristic length l_e explicitely appears in the solution in the elastic zone, whereas the solution in the plastic zone is driven by length l_p. Classical solutions are retrieved for vanishing l_e and l_p.

The use of a single coupled yield criterion (2) leads to non–homogeneous distribution of force and couple stress in the plastic zone, as can be seen from figure 3.

A two–dimensional layer of Cosserat material with infinite extension in direction 1 and height h is considered on figure 2. The unknowns of the problem are $\underline{\boldsymbol{u}} = [u(x_2), 0, 0]^T$ and $\underline{\boldsymbol{\Phi}} = [0, 0, \Phi(x_2)]^T$. Various types of boundary conditions are possible. For example, we consider :

$$u(0) = 0, \Phi(0) = 0, \ \underline{\boldsymbol{t}} = \sigma_{12} \underline{\boldsymbol{e}}_1 = 0, \ \underline{\boldsymbol{m}} = \mu_{32} \, \underline{\boldsymbol{e}}_3 = \mu_{32}^0 \underline{\boldsymbol{e}}_3 \tag{8}$$

Note that the solution of this problem for the classical Cauchy continuum would be a vanishing u. The material exhibits an elastoplastic behaviour with a generalized von Mises yield function (2). Let us recall the elasticity relations in the isotropic case :

$$\underset{\sim}{\boldsymbol{\sigma}} = \lambda(\text{trace}\, \underset{\sim}{\boldsymbol{e}}^e)\, \underset{\sim}{\mathbf{1}} + 2\mu \underset{\sim}{\boldsymbol{e}}^{es} + 2\mu_c \underset{\sim}{\boldsymbol{e}}^{ea}, \quad \underset{\sim}{\boldsymbol{M}} = \alpha(\text{trace}\, \underset{\sim}{\boldsymbol{\kappa}}^e)\, \underset{\sim}{\mathbf{1}} + 2\beta \underset{\sim}{\boldsymbol{\kappa}}^{es} + 2\gamma \underset{\sim}{\boldsymbol{\kappa}}^{ea} \tag{9}$$

where λ, μ are the Lamé constants and $\mu_c, \alpha, \beta, \gamma$ are additional moduli. One usually takes $\beta = \gamma$ at least in the two–dimensional case [Borst, 1991]. An elastic Cosserat characteristic length $l_e = \sqrt{\beta/\mu}$ can be defined. Under the prescribed boundary conditions, a plastic zone develops starting from the top.

Elastic zone, $0 \leq x_2 \leq \alpha$

The combination of elasticity law and balance equations leads to the following equations :

$$\sigma_{12} = (\mu + \mu_c) u_{,2} + 2\mu_c \Phi, \quad \sigma_{21} = (\mu - \mu_c) u_{,2} - 2\mu_c \Phi, \quad M_{32} = 2\beta \Phi_{,2} \tag{10}$$

$$\sigma_{12,2} = 0, \quad M_{32,2} + \sigma_{21} - \sigma_{12} = 0 \tag{11}$$

from which two differential equations are deduced :

$$\Phi_{,22} = \omega_e^2 \Phi, \quad u_{,2} = -\frac{2\mu_c}{\mu + \mu_c} \Phi, \quad \omega_e = \sqrt{\frac{2\mu\mu_c}{\beta(\mu + \mu_c)}} \tag{12}$$

Taking the boundary conditions at the bottom into account, the solutions follow, including an integration constant B to be determined :

$$\Phi(x_2) = B \sinh(\omega_e x_2), \quad u(x) = \frac{2\mu_c B}{\omega_e(\mu + \mu_c)}(1 - \cosh(\omega_e x_2)) \tag{13}$$

$$M_{32} = 2B\beta\omega_e \cosh(\omega_e x_2), \quad \sigma_{21} = -\frac{4\mu\mu_c}{\mu + \mu_c} B \sinh(\omega_e x_2) \tag{14}$$

Plastic zone, $\alpha \leq x_2 \leq h$

In the generalized von Mises criteria (2), the simplifying assumption $a_1 = a, a_2 = 0, b_1 = b, b_2 = 0$ is adopted, together with a constant threshold $R = R_0$. The yield criterion (2) requires :

$$a\sigma_{21}^2 + bM_{32}^2 = R_0^2 \tag{15}$$

Combining this condition with balance equations (11), the solution takes the following form including integration constants C and D :

$$M_{32} = C \cos(\omega_p x_2) + D \sin(\omega_p x_2), \quad \sigma_{21} = \omega_p (C \sin(\omega_p x_2) - D \cos(\omega_p x_2)) \tag{16}$$

$$\omega_p = \frac{1}{l_p} = \sqrt{\frac{b}{a}} \tag{17}$$

where a charateristic length l_p comes into play. The constants C and D are solutions of the following system of equations :

$$C^2 + D^2 = \frac{R_0^2}{b}, \quad C \cos(\omega_p h) + D \sin(\omega_p h) = M_{32}^0 \tag{18}$$

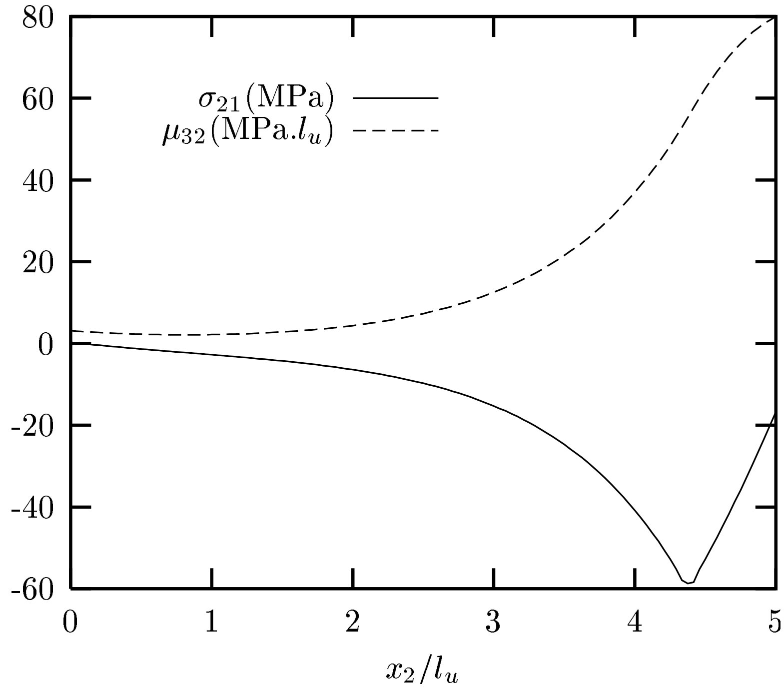

FIG. 3 – Simple glide test for a single criterion von Mises elastoplastic Cosserat infinite layer : force stress and couple stress profiles along a vertical line. A micro–rotation $\Phi = 0.001$ is prescribed at the top $h = 5l_u$. The material parameters are : E = 200000 MPa, $\nu = 0.3$, μ_c=100000 MPa, β=76923 MPa.l_u^2, R_0=100MPa, $a_1 = 1.5, a_2 = 0, b_1 = 1.5l_u^{-2}, b_2 = 0$. The micro–couple prescribed at the top is $M_{32}^0 = 80$MPa.l_u. l_u is a length unit.

The continuity of surface couple vector and yield condition at $x_2 = \alpha$ provides the system of equations for the unknowns B and α :

$$2\beta\omega_e B\cosh(\omega_e\alpha) = C\cos(\omega_p\alpha) + D\sin(\omega_p\alpha) \tag{19}$$

$$16a\left(\frac{\mu\mu_c}{\mu+\mu_c}\right)^2 B^2\sinh^2(\omega_e\alpha) + 4b\beta^2\omega_e^2 B^2\cosh^2(\omega_e\alpha) = R_0^2 \tag{20}$$

The numerical resolution of both systems of equations leads to a semi–analytical solution of the simple glide test, that can be used as test for the implementation of Cosserat elastoplasticity in a Finite Element code. This has been checked for the simulation presented on figure 3.

Références

[Besdo, 1974] Besdo D. (1974). *Ein Beitrag zur nichtlinearen Theorie des Cosserat-Kontinuums.* Acta Mechanica, vol. 20, pp 105–131.

[Borst, 1991] Borst R. de (1991). *Simulation of strain localization : a reappraisal of the Cosserat continuum.* Engng Computations, vol. 8, pp 317–332.

[Borst, 1993] Borst R. de (1993). *A generalization of J_2-flow theory for polar continua.* Computer Methods in Applied Mechanics and Engineering, vol. 103, pp 347–362.

[Koiter, 1963] Koiter W.T. (1963). *Couple-stresses in the theory of elasticity. I and II.* Proc. K. Ned. Akad. Wet., vol. B67, pp 17–44.

[Lippmann, 1969] Lippmann H. (1969). *Eine Cosserat-Theorie des plastischen Fließens*. Acta Mechanica, vol. 8, pp 255–284.

[Mühlhaus and Vardoulakis, 1987] Mühlhaus H.B. and Vardoulakis I. (1987). *The thickness of shear bands in granular materials*. Géotechnique, vol. 37, pp 271–283.

[Nowacki, 1986] Nowacki W. (1986). *Theory of asymmetric elasticity*. Pergamon.

[Sawczuk, 1967] Sawczuk A. (1967). *On yielding of Cosserat continua*. Archives of Mechanics, vol. 19, pp 3–19.